AN ALTERNATIVE DYNAMIC STIFFNESS MATRIX OF AN AXIALLY MOVING STRING

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Summary
The explicit exact dynamic stiffness matrix of an axially moving string element under constant tension has been derived based on the assumption that the material particles enter and leave the domain tangentially. The eigenvalues of an axially moving string in contact with a stationary spring-mass-damper system obtained by using the present element have been compared with the ones obtained by using the element based on the assumption that the material particles enter and leave the domain horizontally. Within wide range of parameters the results are very close to each other. However, in a forced vibration case the results differ considerably.

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1. INTRODUCTION

Axially moving string problems are concerned with vibration and stability of systems where the system mass is in a state of translation and which have negligible bending stiffness. Typical engineering applications of this kind of systems are in the transport of paper sheets, band saw blades, magnetic tapes and power transmission chains. The work done within the field has been widely reviewed by Wickert&Mote [1] and Arbate [2].

Different assumptions concerning the inflow and outflow of the material particles into and out from the domain under consideration lead in axially moving material problems generally to boundary conditions differing from each other which in turn causes differences in the resulting vibration and stability behaviour as has been obtained in the case of transverse oscillations of a fluid conveying pipe [3], [4], [5] and [6]. In axially moving string problems, also, there are more possibilities in defining the boundary conditions than in the case of a stationary string.

Le-Ngoc and McCallion [7] have provided the explicit dynamic stiffness matrix for the transverse oscillations of an axially moving string under constant axial tension. The explicit dynamic stiffness matrix was derived based on the exact solution of the equation of small amplitude motion with boundary conditions based on the assumption that the material particles enter and leave the horizontal domain horizontally.

In practice the supporting rolls have finite radius implying that during the vibration the contact points, i.e. the points where the boundary conditions are specified, wander along the surface of the rolls. Thus, the practical case leads to a nonlinear open boundary problem. Koivurova [8] has considered this problem by using the FEM and among other results obtained that when the amplitude/span ratio is less than 0.005 the differences in the dynamic behaviour between the extreme cases roll radius r=0 and r equal to half of the span are negligible.

The aim of the present article is to provide the explicit and exact dynamic stiffness matrix for the small amplitude vibration of an axially moving string , but now based on the assumption that the material particles enter and leave the domain tangentially. This assumption is physically justified by the fact that every string or band used in practice have small but finite bending stiffness which, when deriving the equation of the motion, can be neglected as a small quantity compared with the geometric stiffness. The differences in the dynamic behaviour of the present model and the model presented in [7] are illustrated by considering the free vibrations of a system comprising of a moving string and a stationary spring-mass-damper system and by comparing the response in a forced vibration case.

2. THEORY

We consider the transverse motion *v(x,t)* of a uniform, flexible string of mass per unit length *m*, under a constant tension *R*, travelling along the *x*-axis with constant speed *c*. The equation of motion reads

$$\frac{\partial^2 v}{\partial t^2}+2c\frac{\partial^2 v}{\partial x \partial t}-(a^2-c^2)\frac{\partial^2 v}{\partial x^2}=0 \tag{1}$$

where

$$a^2=\frac{R}{m} \tag{2}$$

By assuming a separable solution $v(x,t)=V(x)e^{i\omega t}$ for (1) we obtain (for details, see [9])

$$V(x)=B_1 e^{\frac{i\omega x}{a-c}}+B_2 e^{\frac{-i\omega x}{a+c}} \tag{3}$$

There are two natural choices concerning the assumption on the inflow and outflow of the particles, either tangential or horizontal(see Figure 1).

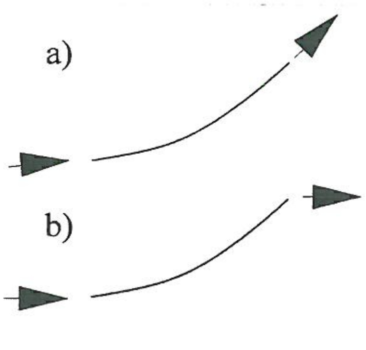

Figure 1. Basic alternatives for the inflow and outflow of particles.

In the former case the boundary condition at the right end is

$$F=ma^2\frac{\partial v}{\partial x}=R\frac{\partial v}{\partial x} \tag{4}$$

and in the latter case

$$F = m(-c\frac{\partial v}{\partial t} + (a^2 - c^2)\frac{\partial v}{\partial x}) \tag{5}$$

respectively ,where F is the external transverse force. At the left end the right hand sides are otherwise of the same form but have opposite signs. The additional terms in (5) are due to the change of the direction of the momentum flux.

By assuming harmonically varying transverse forces at both ends of an element of length L

$$\begin{aligned} p_1 &= P_1 e^{i\omega t} \\ p_2 &= P_2 e^{i\omega t} \end{aligned} \tag{6}$$

with positive direction to the direction of positive v the boundary conditions read

$$\begin{aligned} P_1 e^{i\omega t} &= -R\left[\frac{dV(x)}{dx}\right]_{x=0} e^{i\omega t} \\ P_2 e^{i\omega t} &= R\left[\frac{dV(x)}{dx}\right]_{x=L} e^{i\omega t} \end{aligned} \tag{7}$$

By substituting (3) into (7) we obtain

P=DB **(8)**

Where **P** is the nodal force amplitude vector with elements P_1 and P_2, vector **B** contains coefficients B_1 and B_2 and

$$\mathbf{D} = Ri\omega \begin{bmatrix} -\frac{1}{a-c} & \frac{1}{a+c} \\ \frac{1}{a-c} e^{\frac{i\omega L}{a-c}} & -\frac{1}{a+c} e^{-\frac{i\omega L}{a+c}} \end{bmatrix} \tag{9}$$

which is now naturally much simpler than the corresponding matrix in [7].

The nodal displacement amplitude vector **V** with elements V_1 and V_2 can be written in terms of **B** by using equation (3) as

V=CB (10)

where

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ e^{\frac{i\omega L}{a-c}} & e^{-\frac{i\omega L}{a+c}} \end{bmatrix} \tag{11}$$

By combining equations (8) and (10) we obtain

P=DC^{-1} V=KV (12)

Where **K** is now the dynamic stiffness matrix which can be simplified into the form

$$\mathbf{K}=R\begin{bmatrix} \frac{a\omega}{a^2-c^2}\cot\frac{\omega La}{a^2-c^2}-\frac{i\omega c}{a^2-c^2} & -\frac{a\omega}{a^2-c^2}\csc\frac{\omega La}{a^2-c^2}e^{\frac{-i\omega Lc}{a^2-c^2}} \\ -\frac{a\omega}{a^2-c^2}\csc\frac{\omega La}{a^2-c^2}e^{\frac{i\omega Lc}{a^2-c^2}} & \frac{a\omega}{a^2-c^2}\cot\frac{\omega La}{a^2-c^2}+\frac{i\omega c}{a^2-c^2} \end{bmatrix} \tag{13}$$

The corresponding result of reference [7] is

$$\mathbf{K}=R\begin{bmatrix} \frac{\omega}{a}\cot\frac{\omega La}{a^2-c^2} & -\frac{\omega}{a}\csc\frac{\omega La}{a^2-c^2}e^{\frac{-i\omega Lc}{a^2-c^2}} \\ -\frac{\omega}{a}\csc\frac{\omega La}{a^2-c^2}e^{\frac{i\omega Lc}{a^2-c^2}} & \frac{\omega}{a}\cot\frac{\omega La}{a^2-c^2} \end{bmatrix} \tag{14}$$

Thus, if both ends of the moving string are supported and we do not have any intermediate spring-mass-damper system present both of the dynamic stiffness matrices will lead to identical eigenvalues, because they are obtained from the condition

$$|\mathbf{K}_g|=0 \tag{15}$$

where $\mathbf{K}_g$ is the assembled global dynamic stiffness matrix of the system consisting of two elements necessary due to the zero displacements at the ends.

3. ILLUSTRATIVE EXAMPLES

Le-Ngoc and MacCallion [7] provided eigenvalues of an axially moving string of length L in contact with a stationary load system consisting of a point mass M supported by a spring, k, and a viscous damper, d, for various parameter combinations (see Figure 2). The string is subjected to a tension R to the right of x=A and R-f to the left due to the longitudinal friction force. In what follows nondimensional parameters

$$\omega^*=\omega\frac{L}{a}, v^*=\frac{c}{a}, \xi=\frac{A}{L}, k^*=k\frac{L}{ma^2}, d^*=\frac{d}{ma}, m^*=\frac{M}{mL}, f^*=\frac{f}{R}, \lambda=i\omega^*. \tag{16}$$

are used.

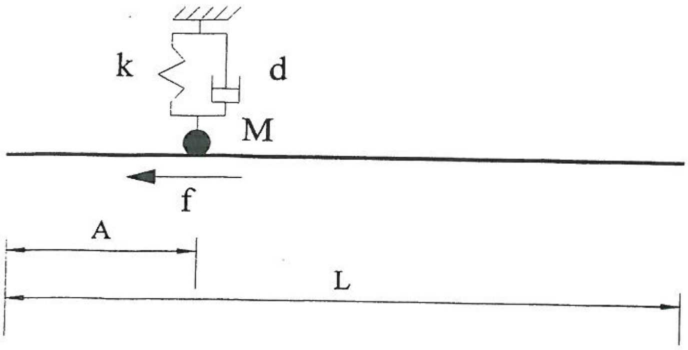

Figure 2. A moving string in contact with a stationary spring-mass-damper system.

The present dynamic stiffness matrix by employing two elements was used to obtain dimensionless eigenvalues for the following six cases

- case a: no intermediate support
- case b: $k^*=20$ at $\xi=0.3$
- case c: $k^*=1000$ at $\xi=0.3$
- case d: $k^*=3$ and $m^*=0.2$ at $\xi=0.3$
- case e: $d^*=1$ at $\xi=0.3$
- case f: $k^*=20$ and $f^*=0.75$ at $\xi=0.5$

For case a identical eigenvalues were obtained by using either of the dynamic stiffness matrices. For cases b...f the results are compared in tables I and II. In cases b...e two elements are used and in case f four elements. The differences between the results are remarkably small in cases b...e.

In case f the tension is in the first span one quarter of that in the second span which lowers the dimensionless critical velocity to $v^*=0.5$. The present results clearly show the instability in that the real part of the eigenvalues become positive above the critical velocity.

Table I

First three dimensionless eigenvalues λ of a moving string in contact with a stationary spring-mass-damper system.

v	Case b k*=20 ξ=0,3 present	Case b k*=20 ξ=0,3 ref.7	Case c k*=1000 ξ=0,3 present	Case c k*=1000 ξ=0,3 ref.7	Case d k*=3,m=0,2 ξ=0,3 present	Case d k*=3,m=0,2 ξ=0,3 ref.7
0	4,2097i	4,2097I	4,4816i	4,4816i	3,2971i	3,2971i
	8,0633i	8,0633I	8,9629i	8,9629i	5,7412i	5,7712i
	9,6569i	9,6569I	10,4374i	10,4374i	9,3042i	9,3042i
0,2	4,0413i	4,0505I	4,3023i	4,3026i	3,1880i	3,1932i
	7,7407i	7,7733I	8,6044i	8,6049i	5,5917i	5,5783i
	9,2706i	9,2802I	10,0199i	10,0212i	8,9409i	8,9373i
0,4	3,5362i	3,5687I	3,7645i	3,7654i	2,8442i	2,8718i
	6,7731i	6,8913I	7,5289i	7,5306i	5,0300i	4,9940i
	8,118i	8,1494I	8,7674i	8,7720i	7,8463i	7,8351i
0,6	2,6942i	2,7520I	2,8682i	2,8697i	2,2231i	2,2996i
	5,1605i	5,3738I	5,7363i	5,7393i	4,0024i	3,9938i
	6,1804i	6,2613I	6,6799i	6,6879i	6,0059i	5,9928i
0,8	1,5155i	1,5761I	1,6134i	1,6148i	1,2793i	1,4033i
	2,9028i	3,1241I	3,2267i	3,2297i	2,3581i	2,4967i
	3,4765i	5,5933I	3,7575i	3,7654i	3,3955i	3,4000i

TABLE II

First three dimensionless eigenvalues λ of a moving string in contact with a stationary spring-mass-damper system.

v^*	Case e $d^*=1$ $\xi=0,3$		Case f $k^*=20$ $f^*=0,75$ $\xi=0,5$	
	Present	Ref [7]	Present	Ref [7]
0	-0,6998+3,2990i	-0,6998+3,2991i	3,0655i	3,0655i
	-1,2018+6,1452i	-1,0218+6,1452i	5,6221i	5,622i
	-0,0959+9,4131i	-0,0959+9,4131i	6,2832i	6,2832i
0,2	-0,6415+3,1523i	-0,6718+3,1671i	-0,0002+2,5766i	2,5962i
	-0,9313+5,9126i	-0,9809+5,8993i	-0,0023+5,0472i	5,0803i
	-0,0884+9,0375i	-0,0921+9,0365i	-0,0017+5,6291i	5,6279i
0,4	-0,4839+2,7254i	-0,5878+2,7712i	-0,0008+1,1056i	1,1216i
	-0,6927+5,2027i	-0,8583+5,1619i	-0,0034+2,2092i	2,2423i
	-0,0676+7,9100i	-0,0806+7,9070i	-0,0092+3,3056i	3,3602i
0,6	-0,2754+2,0463i	-0,4478+2,1114i	0,0044+1,3511i	
	-0,3877+3,9905i	-0,6539+3,9329i	0,1194+3,6653i	
	-0,0392+6,0289i	-0,0614+6,0244i	0,0017+4,1261i	
0,8	-0,0855+1,1369i	-0,2519+1,1877i	0,2688+1,9197i	
	-0,1189+2,2568i	-0,3678+2,2123i	0,0465+2,0727i	
	-0,0124+3,3924i	-0,0345+3,3887i	0,2025+2,2829i	

However, when we consider the forced vibrations of an axially moving string whose left end is supported and the right end is subjected to a harmonically varying transverese force having amplitude P_2 the effect of the imaginary part in the dynamic stiffness matrix is clearly seen. Figure 3 shows the dimensionless transverse vibration amplitude $V^*= |\ V_2\ R/\ P_2L|$ as a function of the dimensionless angular frequency ω^*. The location of the resonance peaks correspond to the natural angular frequencies

$$\omega_n^* = \frac{(2n-1)\pi}{2}(1-v^{*2}) \qquad (14)$$

derived in reference [9].

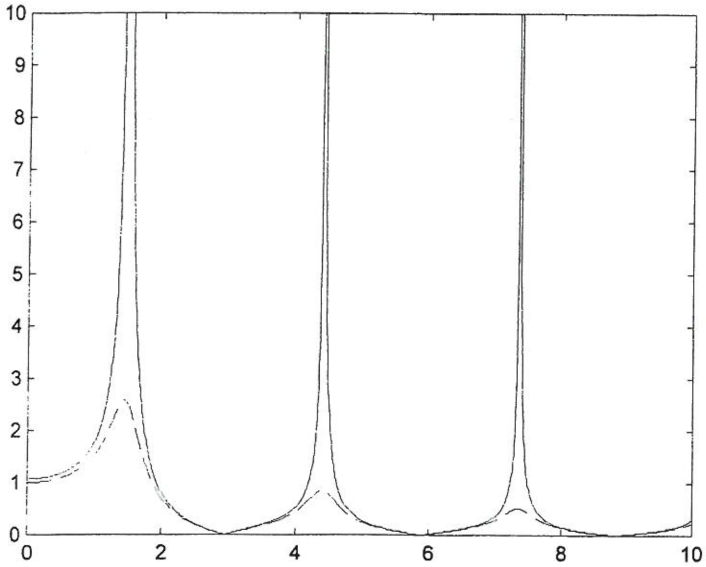

Figure 3. Dimensionless vibration amplitude V^* as function of dimensionless angular frequency ω^* at v*=0.25; ____ horizontal exit, ----tangential exit.

The Coriolis force has a strong damping effect when tangential exit is assumed and that the solution equals the static one when ω^*=0. When horizontal exit is assumed a value larger than one is obtained for the dimensionless vibration amplitude at ω *=0 due to the change in the direction of the momentum flux. When the direction of the axial motion is reversed, the model based on horizontal enter predicts neutral stability, i.e. purely imaginary eigenvalues but the tangential enter leads to complex eigenvalues having positive real parts indicating instability when the axial velocity is not equal to zero.

4. CONCLUSIONS

The explicit dynamic stiffness matrix of an axially moving string under constant tension has been derived based on the assumption that the material particles enter and leave the domain tangentially. The eigenvalues of an axially moving string in contact with a stationary spring-mass-damper system obtained have been compared with the ones based on the assumption that the material particles enter and leave the domain horizontally[7]. Within wide range of parameters the results are very close to each other. Moreover, the differences between the two matrices are illustrated by considering a forced vibration case where the results differ considerably.

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