# ON THE WEIGHT FACTORS IN THE LEAST SQUARES FINITE ELEMENT METHOD

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## ABSTRACT

It seems that no general rules have been given in the literature how to select the relative weights for the different field equation residuals when the least squares method is employed in the finite element method. This article suggest a possible strategy for this employing the one-dimensional heat conduction problem as an illuminating demonstration case. The goal is to achieve the nodally exact solution. Local solution behavior is employed in a version of the patch test to determine the weight factors. In this simple case the optimum weight factors are found to behave quite unexpectedly.

## HEAT CONDUCTION PROBLEM

Let us consider the problem described by the field equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(-k\frac{\mathrm{d}T}{\mathrm{d}x})-s=0, \qquad 0<x<L \tag{1}$$

and the boundary conditions

$$T=\overline{T}, \qquad x=0 \tag{2}$$

$$-k\frac{\mathrm{d}T}{\mathrm{d}x}=\overline{q}, \quad x=L. \tag{3}$$

$T(x)$ is the temperature to be determined as a function of the independent variable $x$. The thermal conductivity $k(x)$ and the heat source rate per volume $s(x)$ are given quantities. The boundary conditions mean that on the left hand boundary the value of the temperature is given $=\overline{T}$ and that on the right hand boundary the value of the heat flux is given $=\overline{q}$ (Figure 1).

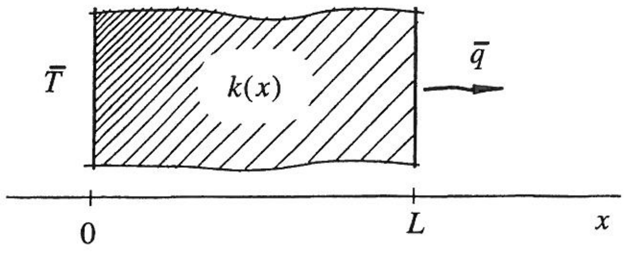


**Figure 1** One-dimensional heat conduction through a wall.

To deal with a $C^0$-continuous finite element approximation using the least squares method we define a new unknown variable (the heat flux)

$$q = -k\frac{\mathrm{d}T}{\mathrm{d}x} \tag{4}$$

and describe the problem now by the field equations

$$R_1 \equiv \frac{\mathrm{d}q}{\mathrm{d}x} - s = 0, \qquad 0 < x < L, \tag{5}$$

$$R_2 \equiv q + k\frac{\mathrm{d}T}{\mathrm{d}x} = 0, \quad 0 < x < L, \tag{6}$$

and by the boundary conditions

$$T = \bar{T}, \qquad x = 0, \tag{7}$$

$$q = \bar{q}, \qquad x = L. \tag{8}$$

**Remark 1.** Another obvious quantity as a new variable could be directly the derivative $\mathrm{d}T/\mathrm{d}x$. However, with a discontinuous thermal conductivity — for instance in connection with a composite material wall — the derivative has jumps at the material interfaces but the flux is continuous. Thus as we use here continuous approximations, the selection of $q$ as the new variable is more appropriate. □

## LEAST SQUARES FORMULATION

The least squares expression corresponding to (5) and (6) can be written using obvious matrix notation as

$$\Pi(T,q) = \frac{1}{2}\int_0^L \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \mathrm{d}x. \tag{9}$$

The boundary conditions (7) and (8) are assumed to be satisfied in advance by the admissible $T$ and $q$ in (9). The weight factor matrix $[\alpha]$ can be taken symmetric ($\alpha_{12} = \alpha_{21}$) without loss of generality. The elements of the matrix must naturally have such physical dimensions that the least squares expression is dimensionally homogeneous. The question is: How can one select the elements of the weight factor matrix in a logical way when the finite element method is used? It is obvious that the values of the weightings have in general an effect on the discrete solution. It seems that in the literature the values are taken based just on numerical experiments.

As only the relative values of the weight factors are of importance we simplify the notation by taking $\alpha_{11} = 1$ and by denoting $\alpha_{12} = \alpha_{21} = \beta$ and $\alpha_{22} = \alpha$. Expression (9) is now in detail

$$\Pi(T,q) = \frac{1}{2}\int_0^L [R_1^2 + 2\beta R_1 R_2 + \alpha R_2^2]\mathrm{d}x. \tag{10}$$

## FINITE ELEMENT APPROXIMATION

We employ two-noded linear element approximation — shown schematically in Figure 2 (a) — for $T$ and $q$:

$$\begin{aligned} \tilde{T}(x) &= \sum_j N_j(x) T_j, \\ \tilde{q}(x) &= \sum_j N_j(x) q_j, \end{aligned} \tag{11}$$

where the $N_j$ are the global shape functions and $T_j$ and $q_j$ are the nodal values of the approximations.

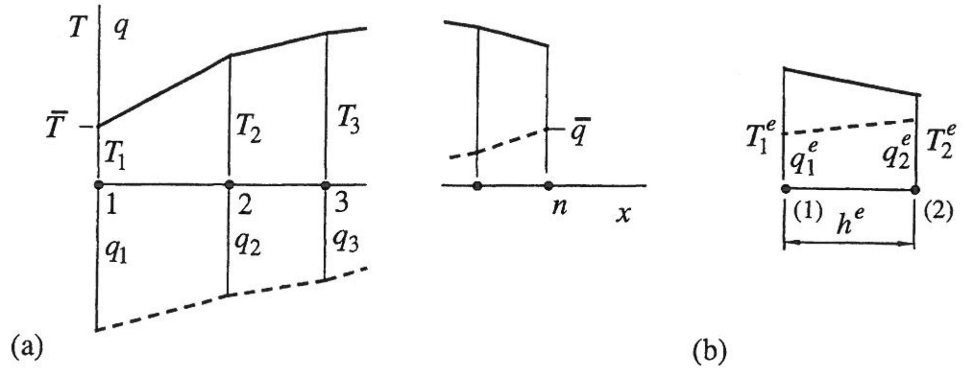


**Figure 2 (a)** Approximations. **(b)** Approximations in an element.

Approximations (11) are substituted in (10) and the system equations are obtained from the stationarity conditions

$$\frac{\partial \tilde{\Pi}}{\partial T_i} = 0, \qquad \frac{\partial \tilde{\Pi}}{\partial q_i} = 0 \qquad (12)$$

with $i = 1,2,\cdots,n$. We develop these formulas further in some detail. The field equation residuals are

$$\tilde{R}_1 = \Sigma N_j' q_j - s, \qquad \tilde{R}_2 = \Sigma N_j q_j + k \Sigma N_j' T_j \qquad (13)$$

with the obvious meaning for the dash notation. We obtain

$$\begin{aligned}
\frac{\partial \tilde{\Pi}}{\partial T_i} &= \int_0^L [\tilde{R}_1 \frac{\partial \tilde{R}_1}{\partial T_i} + \beta \tilde{R}_1 \frac{\partial \tilde{R}_2}{\partial T_i} + \beta \tilde{R}_2 \frac{\partial \tilde{R}_1}{\partial T_i} + \alpha \tilde{R}_2 \frac{\partial \tilde{R}_2}{\partial T_i}] \mathrm{d}x \\
&= \int_0^L [0 + \beta(\Sigma N_j' q_j - s) k N_i' + 0 + \alpha(\Sigma N_j q_j + k \Sigma N_j' T_j) k N_i'] \mathrm{d}x \\
&= \Sigma (\int_0^L \alpha k^2 N_i' N_j' \mathrm{d}x) T_j + \\
&+ \Sigma [\int_0^L (\alpha k N_i' N_j + \beta k N_i' N_j') \mathrm{d}x] q_j + \\
&- \int_0^L \beta k N_i' s \mathrm{d}x, \\
\frac{\partial \tilde{\Pi}}{\partial q_i} &= \int_0^L [\tilde{R}_1 \frac{\partial \tilde{R}_1}{\partial q_i} + \beta \tilde{R}_1 \frac{\partial \tilde{R}_2}{\partial q_i} + \beta \tilde{R}_2 \frac{\partial \tilde{R}_1}{\partial q_i} + \alpha \tilde{R}_2 \frac{\partial \tilde{R}_2}{\partial q_i}] \mathrm{d}x \\
&= \int_0^L [(\Sigma N_j' q_j - s) N_i' + \beta(\Sigma N_j' q_j - s) N_i + \\
&\qquad + \beta(\Sigma N_j q_j + k \Sigma N_j' T_j) N_i' + \alpha(\Sigma N_j q_j + k \Sigma N_j' T_j) N_i] \mathrm{d}x \\
&= \Sigma [\int_0^L (\alpha k N_i N_j' + \beta k N_i' N_j') \mathrm{d}x] T_j + \\
&+ \Sigma [\int_0^L (N_i' N_j' + \alpha N_i N_j + \beta(N_i N_j' + N_i' N_j)) \mathrm{d}x] q_j + \\
&- \int_0^L (N_i' + \beta N_i) s \mathrm{d}x.
\end{aligned} \qquad (14)$$

The discrete system can be written thus as

$$\sum_{j=1}^{n} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{ij} \begin{Bmatrix} T \\ q \end{Bmatrix}_j - \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}_i = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad i = 1,2,\cdots,n \qquad (15)$$

with

$$
\begin{aligned}
(K_{11})_{ij} &= \int_0^L \alpha k^2 N_i' N_j' \,\mathrm{d}x, \\
(K_{12})_{ij} &= \int_0^L (\alpha k N_i' N_j + \beta k N_i' N_j')\mathrm{d}x, \\
(K_{21})_{ij} &= \int_0^L (\alpha k N_i N_j' + \beta k N_i' N_j')\mathrm{d}x, \\
(K_{22})_{ij} &= \int_0^L (N_i' N_j' + \alpha N_i N_j + \beta(N_i N_j' + N_i' N_j))\mathrm{d}x, \\
(b_1)_i &= \int_0^L \beta k N_i' s \mathrm{d}x, \\
(b_2)_i &= \int_0^L (N_i' + \beta N_i) s \mathrm{d}x.
\end{aligned}
\tag{16}
$$

In what follows we assume constant $\alpha$ and $\beta$ in an element; of course the values can be different from element to element. Assuming further for simplicity some constant representative value for $k$ in an element the members of the element stiffness matrix are easily evaluated in closed form. Using the local element node numbering of Figure 2 (b), the element contribution expressions are

$$
\sum_{j=1}^{2} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{ij}^{e} \begin{Bmatrix} T \\ q \end{Bmatrix}_j^{e} - \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}_i^{e}, \quad i = 1,2 \tag{17}
$$

with

$$
\begin{aligned}
(K_{11})_{ij} &= \frac{\alpha k^2}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\
(K_{12})_{ij} &= \frac{\alpha k}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} + \frac{\beta k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\
(K_{21})_{ij} &= \frac{\alpha k}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{\beta k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\
(K_{22})_{ij} &= \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\alpha h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \beta \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\
(b_1)_i &= \beta k s \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \\
(b_2)_i &= s \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} + \frac{\beta h s}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.
\end{aligned}
\tag{18}
$$

The source term contributions are evaluated here just for a constant $s$. The sixteen members of the element stiffness matrix and the four members of the "load vector" are given in a condensed matrix form with index $i$ referring to the row number and $j$ to the column number. The element index $e$ has been left out for simplicity.

## A LOGIC FOR THE SELECTION OF THE WEIGHT FACTORS

The weighted residual formulations — the least squares method can also be considered as a weighted residual formulation, e.g. [1] — in a way give up the study of the detailed field equations and consider them only in an average, integrated sense. We now try to inject information of the actual local solution behaviour into the formulation. Let us consider a generic point in the domain of the solution. To simplify the treatment we take the operator data (here $k$) to have some constant representative values. (See Remarks 3 and 4). Field equation (1) obtains the form

$$-k_0 \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} - (s_0 + s_0' x + \tfrac{1}{2} s_0'' x^2 + \tfrac{1}{6} s_0''' x^3 + \cdots) = 0. \tag{19}$$

Without loss of generality the local origin has been taken at the point under study. The source term has been developed into a Taylor series. Equation (19) is a linear second order constant coefficient differential equation. Its analytical solution can be obtained by the well known methods explained in mathematics texts. The solution is

$$T = A + Bx - \frac{1}{2}\frac{s_0}{k_0} x^2 - \frac{1}{6}\frac{s_0'}{k_0} x^3 - \frac{1}{24}\frac{s_0''}{k_0} x^4 - \frac{1}{120}\frac{s_0'''}{k_0} x^5 - \cdots \tag{20}$$

where $A$ and $B$ are the two integration constants. Thus the behaviour of the solution in the neigborhood of the point under study must be approximately of the form (20). How can we make use of this information? At a first glance it seems that some kind of iterative procedure would be needed as the values of the integration constant should have to be found at the point under question from a preliminary numerical solution. Fortunately this is not the case. It is presently found that the values of the integration constants are not needed at all.

The flux from (20) is

$$q = -k_0 B + s_0 x + \frac{1}{2} s_0' x^2 + \frac{1}{6}\frac{s_0''}{k_0} x^3 + \frac{1}{24}\frac{s_0''}{k_0} x^4 + \cdots \tag{21}$$

We collect $T$, $q$ and $s$ in a column vector and obtain

$$\begin{Bmatrix} T \\ q \\ s \end{Bmatrix} = A\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + B\begin{Bmatrix} x \\ -k_0 \\ 0 \end{Bmatrix} + s_0\begin{Bmatrix} -1/(2k_0)\cdot x^2 \\ x \\ 1 \end{Bmatrix} + s_0'\begin{Bmatrix} -1/(6k_0)\cdot x^3 \\ 1/2\cdot x^2 \\ x \end{Bmatrix} +$$

$$+ s_0''\begin{Bmatrix} -1/(24k_0)\cdot x^4 \\ 1/6\cdot x^3 \\ 1/2\cdot x^2 \end{Bmatrix} + s_0'''\begin{Bmatrix} -1/(120k_0)\cdot x^5 \\ 1/24\cdot x^4 \\ 1/6\cdot x^3 \end{Bmatrix} + \cdots \qquad (22)$$

We call this combination of $T$, $q$ and $s$ as the reference solution. The values $A$, $B$, $s_0$, $\cdots$ fix the solution. We obtain specific reference solutions by taking consecutively only $A \neq 0$, only $B \neq 0$, only $s_0 \neq 0$, $\cdots$. These give the reference solutions (see Remark 2)

$$\begin{aligned} &T = 1, \quad q = 0, \quad s = 0, \\ &T = x, \quad q = -k_0, \quad s = 0, \\ &T = -\frac{1}{2}\frac{x^2}{k_0}, \quad q = x, \quad s = 1, \\ &\ldots \end{aligned} \qquad (23)$$

**Remark 2.** It is found that $A$, $B$, $s_0$ , $\cdots$ cancel in the equations used in the patch test so that we can simply take here $A = 1$, $B = 1$, $\cdots$. □

## PATCH TEST

In an ideal finite element calculation we would always get the nodally exact solution. A nodally exact solution is clearly very beneficial for post-processing and adaptive procedures. This ideal can rarely be achieved in reality but we can strive for it. For this purpose we apply the patch test, described e.g. in [2], in a special way making use of the reference solutions.

Consider node $i$ inside a regular mesh at the point under study (Figure 3 (a)).

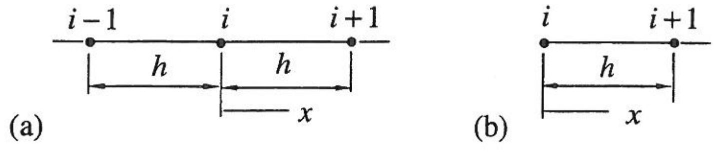


**Figure 3 (a)** Two element patch. **(b)** One element patch.

The two system equations corresponding to node $i$ are

$$
\begin{aligned}
&-\frac{\alpha k^2}{h}T_{i-1}+\frac{2\alpha k^2}{h}T_i-\frac{\alpha k^2}{h}T_{i+1}+\\
&(\frac{\alpha k}{2}-\frac{\beta k}{h})q_{i-1}+\frac{2\beta k}{h}q_i+(-\frac{\alpha k}{2}-\frac{\beta k}{h})q_{i+1}-\beta k\int_{-h}^{h}N_i' s\mathrm{d}x=0,\\
&(-\frac{\alpha k}{2}-\frac{\beta k}{h})T_{i-1}+2\frac{\beta k}{h}T_i+(\frac{\alpha k}{2}-\frac{\beta k}{h})T_{i+1}+\\
&(-\frac{1}{h}+\frac{\alpha h}{6})q_{i-1}+(\frac{2}{h}+\frac{4\alpha h}{6})q_i+(-\frac{1}{h}+\frac{\alpha h}{6})q_{i+1}-\int_{-h}^{h}N_i' s\mathrm{d}x-\beta\int_{-h}^{h}N_i s\mathrm{d}x=0.
\end{aligned}
\tag{24}
$$

The Euler equations following from the stationarity of (10) (with constant $k$, $\alpha$ and $\beta$) are found to be

$$
\begin{aligned}
&-\beta k\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}-\alpha k(\frac{\mathrm{d}q}{\mathrm{d}x}+k\frac{\mathrm{d}^2 T}{\mathrm{d}x^2})+\beta k\frac{\mathrm{d}s}{\mathrm{d}x}=0,\\
&-\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}+\alpha(q+k\frac{\mathrm{d}T}{\mathrm{d}x})-\beta k\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}+\frac{\mathrm{d}s}{\mathrm{d}x}-\beta s=0.
\end{aligned}
\tag{25}
$$

Thus, the discrete equations are rather transparent if we consider the Galerkin method applied on (25).

We now take the nodal values for $T$ and $q$ at nodes $i-1$, $i$, $i+1$ evaluated from the specific reference solutions (23) and demand equations (24) to be valid using the corresponding source term $s$. The Mathematica program is used conveniently to perform the rather tedious calculations. It is found that in the first three cases the equations are satisfied — the patch test is passed — irrespective of the values of $\alpha$ and $\beta$. After that it is found that one must put $\alpha=0$ and $\beta$ can have any non-zero value and the patch test is always passed no matter the series term order! In fact, a general specific reference solution continuing (23) is obtained as (multiplication by a constant for simplicity is performed)

$$
T=-\frac{1}{k_0(n+1)(n+2)}x^{n+2},\quad q=\frac{1}{n+1}x^{n+1},\quad s=x^n. \tag{26}
$$

This produces in (24) with arbitrary $\alpha$ and $\beta$ when for instance $n=100$ the residuals

$$
\begin{aligned}
&-\frac{50\alpha k h^{101}}{5151},\\
&0.
\end{aligned}
\tag{27}
$$

As equations (24) are linear with respect to the nodal values and with respect to the source term, the linear combination (22) with any values of $A$, $B$, $\cdots$ passes the patch test when $\alpha = 0$. Thus we have been able to make use of the local solution without actual knowledge of the local integration constants.

The result obtained is rather suprising as often in the least squares method the weight factor matrix is taken from the outset to be diagonal. Here this selection is seen not to be the best possible.

The calculations are repeated for a patch (one element) at the end of the domain (Figure 3 (b)). Now either temperature or flux is given and correspondingly only either of the system equations for node $i$

$$\begin{aligned}
&\frac{\alpha k^2}{h}T_i - \frac{\alpha k^2}{h}T_{i+1} + \\
&+(-\frac{\alpha k}{2}+\frac{\beta k}{h})q_i + (-\frac{\alpha k}{2}-\frac{\beta k}{h})q_{i+1} - \beta k\int_0^h N_i' s\mathrm{d}x = 0, \\
&(-\frac{\alpha k}{2}+\frac{\beta k}{h})T_i + (\frac{\alpha k}{2}-\frac{\beta k}{h})T_{i+1} + \\
&+(\frac{1}{h}+\frac{\alpha h}{3}-\beta)q_i + (-\frac{1}{h}+\frac{\alpha h}{6})q_{i+1} - \int_0^h N_i' s\mathrm{d}x - \beta\int_0^h N_i s\mathrm{d}x = 0.
\end{aligned} \tag{28}$$

is available. The patch test is performed similarly as above. Now the first two cases (23) are passed automatically but the third and fourth case are found to give the residuals

$$\begin{aligned}
&0, \\
&-\frac{\alpha h^2}{12},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&-\frac{\alpha k h^5}{15}, \\
&\frac{\alpha h}{60},
\end{aligned} \tag{30}$$

respectively. Continuing, it is again found we must put $\alpha = 0$ and $\beta$ can have any non-zero value.

**Remark 3.** It should be noted that the approximation introduced by assuming constant operator data in evaluating the reference solutions does not produce any errors in the least squares formulation as such. This approximation is used only to give us readily some suitable reference solutions to be used in the determination of roughly optimum weighting factors. Obviously nearly any selection of the weighting factors leads finally to a convergent solution when the mesh gets dense enough. However, our goal is to have nodally accurate results already with practical meshes. □

**Remark 4.** A more accurate representation of a reference solution could perhaps be achieved in the case of a varying $k$ by first writing equation (1) as

$$-k\frac{\mathrm{d}^2T}{\mathrm{d}x^2}-\frac{\mathrm{d}k}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}-s=0 \tag{31}$$

and after that taking $k$ and $\mathrm{d}k/\mathrm{d}x$ to have some local constant values. However, based on the discussion in Remark 3, there are hardly grounds to complicate the treatment in this manner. □

## NUMERICAL EXAMPLE

A numerical example case has been considered using dimensionless presentation with the data

$$L=1, \qquad k=1, \qquad s=-\sin(4\pi x), \qquad \overline{T}(0)=0, \qquad \overline{q}(1)=\frac{1}{4\pi}. \tag{32}$$

The exact solution is

$$T=\frac{1}{16\pi^2}[-\sin(4\pi x)]\equiv\frac{1}{16\pi^2}\hat{T}(x), \qquad q=\frac{1}{4\pi}\cos(4\pi x)\equiv\frac{1}{4\pi}\hat{q}(x). \tag{33}$$

The results of some calculations are shown in Figures 4, 5, 6, and 7. The exact distributions are given by the smooth curves. The results are actually expressed for the scaled quantities $\hat{T}$ and $\hat{q}$ defined by formulas (33). Only five elements with a uniform and an irregular mesh are employed.

In Figures 4 and 5 a selection of the optimum weight factors is used. The numerical results are seen to be according to the theory. In Figures 6 and 7 a non-optimum weight factor selection is used. The temperature distribution is now found to be rather poor. It should be mentioned that in the case of a varying conductivity even with the optimum weight factor values no quite nodally exact results for the temperature are any more to be expected.

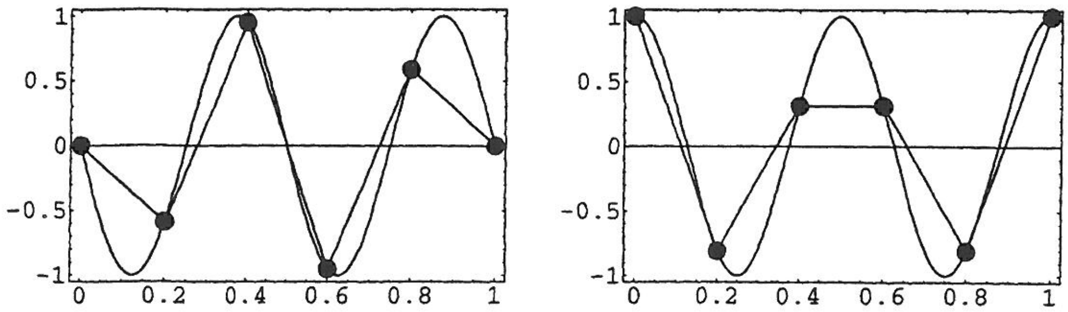


**Figure 4** Weights: $\alpha = 0$ and $\beta = 1$. Left, temperature. Right, heat flux.

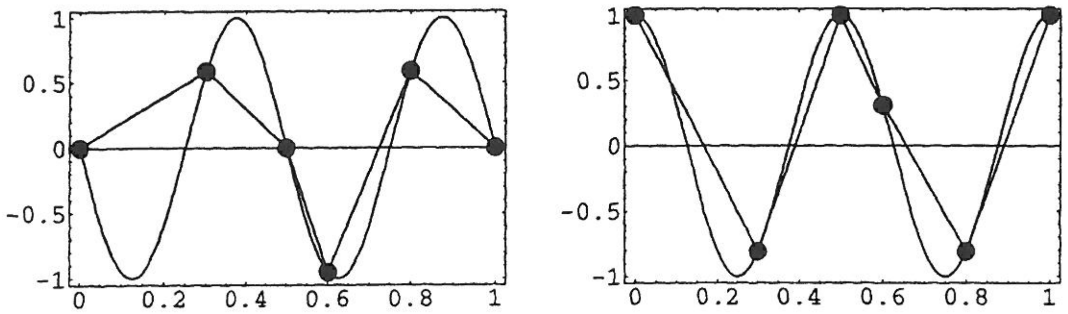

**Figure 5** Weights: $\alpha = 0$ and $\beta = 1$. Left, temperature. Right, heat flux.

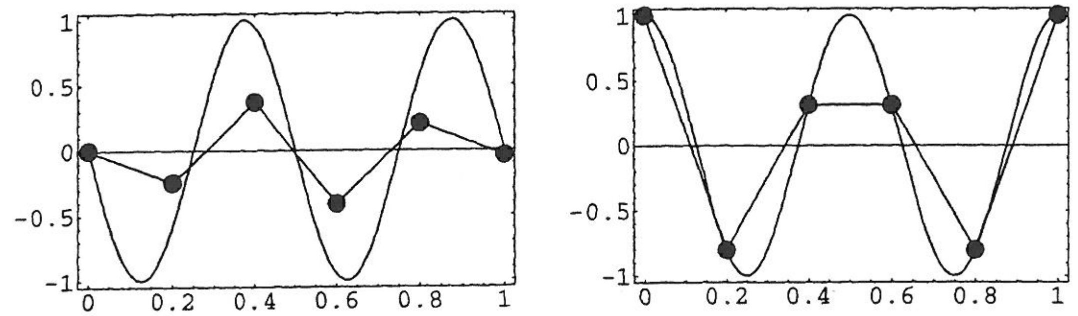

**Figure 6** Weights: $\alpha = 1$ and $\beta = 0$. Left, temperature. Right, heat flux.

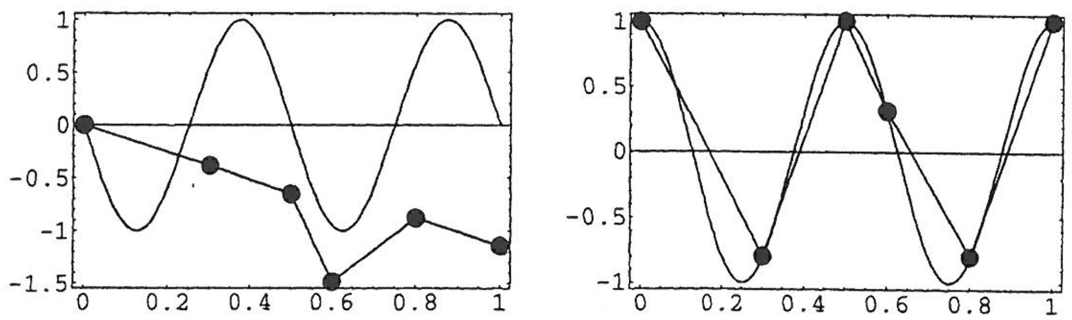


**Figure 7** Weights: $\alpha = 1$ and $\beta = 0$. Left, temperature. Right, heat flux.

## CONCLUDING REMARKS

The initial ideas to determine good weighting factors described above in connection with the one-dimensional heat conduction case can hopefully be generalized to more complicated problems and to more than one dimension although work remains to be done to find the best approaches. It may prove that the goal of exact nodal values can be a too demanding one and some "average" type criteria must be applied. In any case, the possibility to employ reference solutions to inject more information into the formulation may be valuable. The lines of thought used above are similar to those employed for instance in References [3], [4], [5], and [6] with some success for the determination of tuning parameter values in the stabilized or so-called sensitized finite element Galerkin methods. These references describe certain attempts to find reference solutions and to apply the patch test in more than one dimension. In addition to ad hoc procedures extending one-dimensional results, the type of approach described below could be speculated on.

Let us consider as an example case again the heat conduction problem now in two dimensions. The governing field equation for an isotropic material is

$$\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y}\right)-s=0 \tag{34}$$

or using a form more suitable with the least squares method:

$$\begin{aligned}
&\frac{\partial q}{\partial x}+\frac{\partial r}{\partial y}-s=0,\\
&q+k\frac{\partial T}{\partial x}=0,\\
&r+k\frac{\partial T}{\partial y}=0.
\end{aligned} \tag{35}$$

A possibility to find reference solutions could be as follows. Starting from (34), $T$ and $s$ are developed into Taylor series:

$$\begin{aligned}
T = T_0 &+ (T_x)_0 x + (T_y)_0 y + \frac{1}{2}(T_{xx})_0 x^2 + (T_{xy})_0 xy + \frac{1}{2}(T_{yy})_0 y^2 + \\
&+ \frac{1}{6}(T_{xxx})_0 x^3 + \frac{1}{2}(T_{xxy})_0 x^2 y + \frac{1}{2}(T_{xyy})_0 xy^2 + \frac{1}{6}(T_{yyy})_0 y^3 \cdots
\end{aligned} \tag{36}$$

$$s = s_0 + (s_x)_0 x + (s_y)_0 y + \frac{1}{2}(s_{xx})_0 x^2 + (s_{xy})_0 xy + \frac{1}{2}(s_{yy})_0 y^2 + \cdots \tag{37}$$

Assuming a locally constant $k$, we obtain from (34) and from its differentiated forms evaluated at the local origin:

$$\begin{aligned}
&-k_0(T_{xx})_0 - k_0(T_{yy})_0 - s_0 = 0, \\
&-k_0(T_{xxx})_0 - k_0(T_{yyx})_0 - (s_x)_0 = 0, \\
&-k_0(T_{xxy})_0 - k_0(T_{yyy})_0 - (s_y)_0 = 0, \\
&-k_0(T_{xxxx})_0 - k_0(T_{yyxx})_0 - (s_{xx})_0 = 0, \\
&\dots
\end{aligned} \tag{38}$$

These equations contain information about the governing field equation. We obtain, say,

$$\begin{aligned}
&(T_{xx})_0 = -(T_{yy})_0 - \frac{s_0}{k_0}, \\
&(T_{xxx})_0 = -(T_{yyx})_0 - \frac{(s_x)_0}{k_0}, \\
&(T_{yyy})_0 = -(T_{xxy})_0 - \frac{(s_y)_0}{k_0}, \\
&\dots
\end{aligned} \tag{39}$$

Substituting these in (36) gives

$$\begin{aligned}
T &= T_0 + (T_x)_0 x + (T_y)_0 y + \frac{1}{2}[-(T_{yy})_0 - \frac{s_0}{k_0}]x^2 + (T_{xy})_0 xy + \frac{1}{2}(T_{yy})_0 y^2 + \\
&+ \frac{1}{6}[-(T_{yyx})_0 - \frac{(s_x)_0}{k_0}]x^3 + \frac{1}{2}(T_{xxy})_0 x^2 y + \frac{1}{2}(T_{xyy})_0 xy^2 + \\
&+ \frac{1}{6}[-(T_{xxy})_0 - \frac{(s_y)_0}{k_0}]y^3 + \cdots \\
&= T_0 + (T_x)_0 x + (T_y)_0 y + (T_{xy})_0 xy + (T_{yy})_0(-\frac{1}{2}x^2 + \frac{1}{2}y^2) + \\
&+ (T_{xxy})_0(\frac{1}{2}x^2 y - \frac{1}{6}y^3) + (T_{xyy})(-\frac{1}{6}x^3 + \frac{1}{2}xy^2) + s_0(-\frac{1}{2k_0}x^2) + \\
&+ (s_x)_0(-\frac{1}{6k_0}x^3) + (s_y)_0(-\frac{1}{6k_0}y^3) + \cdots
\end{aligned} \tag{40}$$

Ending at this, we have ten independent multipliers and thus ten specific reference solutions available. (The corresponding $p$, $r$ and $s$ are obtained from (35) and (37).) It remains to be studied if this kind of approach will yield any useful results.

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