

# A NOVEL MODEL FOR SPEED SKIING

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## ABSTRACT

A novel model for speed skiing is developed. The model takes into account gravity, friction and air resistance. Moreover, the friction may be velocity dependent. Closed form solutions are obtained for the constant slope case. The results indicate that the effect of aerodynamic lift is negligible when compared to the effect of velocity dependence of the friction.

## INTRODUCTION

The inspiration for the present work have been the numerous speculations on the factors affecting the speed of a skier on a downhill during the World Championships in Winter Sports in Trondheim, Norway.

From the engineering mechanics point of view the only correct explanation can be found by considering the equation of motion of the skier. A simple model was created in [1] in order to explain why a larger skier glides a slope faster than a smaller one. Coulomb friction with constant frictional coefficient and air resistance proportional to the velocity squared were accounted for in the model. Later, Holmlund [2] has refined the model by taking into account also the aerodynamic lift and the effect of wind and solved the equation of motion numerically. When the skier is in the "egg position" the effect of aerodynamic lift is, however, negligible according to Reference [3]. In the model of [2] as in the model given in [1] the coefficient of friction has been assumed to be constant.

Distinctions between static and kinetic friction have been mentioned in the literature for centuries since the work of Euler [4]. Coulomb [5] was obviously the first who observed the dependence of the kinetic friction on the sliding velocity for certain combinations of materials and surface conditions. Measurements done for the plastics-ice combination

indicate that the decrease of friction coefficient with increasing sliding velocity can be up to 50% when the temperature is below  $-10^{\circ}\text{C}$  (see, e.g. [6]). Therefore, velocity dependence is incorporated into the present model. Unpublished material suggests that the linear variation is valid at least within certain range of velocity, [7], when the ski-snow combination is under consideration. Hence, as a first approximation we assume a linear variation.

It appears that explicit solution of the equation of motion is possible. By inspecting the solutions time versus velocity,  $t=t(v)$ , and distance traversed versus velocity,  $s=s(v)$ , some interesting conclusions can be made.

### EQUATION OF MOTION AND SOLUTIONS

From the Newton's second law we obtain two equations of motion (along the direction of the motion and in the direction perpendicular to it). By eliminating the normal force one obtains

$$m \frac{dv}{dt} = m g (\sin \alpha - \mu \cos \alpha) + \mu L - D \quad (1)$$

where  $m$  is the mass of the skier,  $v$  is the velocity,  $g$  is the acceleration of gravity,  $\alpha$  is the (constant) slope of the hill,  $\mu$  is the coefficient of friction,  $L$  is the aerodynamic lift and  $D$  is the aerodynamic drag. In the range of velocities under consideration we can express  $L$  and  $D$  in the form

$$L = C_L \frac{1}{2} \rho (v + v_t)^2 A \quad (2)$$

$$D = C_D \frac{1}{2} \rho (v + v_t)^2 A \quad (3)$$

where  $\rho$  is the density of the air,  $A$  is the reference cross-sectional area,  $v_t$  is the velocity of the wind along the inclined plane (positive when against the motion),  $C_L$  and  $C_D$  are the lift and drag coefficients, respectively. By introducing the non-dimensional time, distance and velocity as follows

$$T = \sqrt{fh}t \quad (4)$$

$$S = sh \quad (5)$$

and

$$V = v \sqrt{\frac{h}{f}} \quad (6)$$

where

$$f = g(\sin \alpha - \mu \cos \alpha) \quad (7)$$

and

$$h = \frac{(C_D - \mu C_L) A \rho}{2m} \quad (8)$$

we get the equation of motion into the form

$$\frac{dV}{dT} = 1 - (V + V_t)^2 \quad (9)$$

where  $V_t$  is the nondimensional wind velocity. From equation (9) we immediately see that the maximum value of the non-dimensional velocity is equal to 1, if the wind velocity equals zero. Hence, the maximum speed attainable is  $\sqrt{f}h$ , see equation (6).

If the chain rule of differentiation is employed, the alternative form

$$V \frac{dV}{dS} = 1 - (V + V_t)^2 \quad (10)$$

is obtained.

If we now assume that

$$\mu = \mu_0(1 - \varepsilon V) \quad (11)$$

where  $\varepsilon$  is a parameter small enough to prevent negative values of  $\mu$ , the nondimensional forms corresponding to (9) and (10) are (from now on  $\mu_0$  replaces  $\mu$  in equation (7))

$$\frac{dV}{dT} = 1 + \varepsilon V - (V + V_t)^2 \quad (12)$$

and

$$V \frac{dV}{dS} = 1 + \varepsilon V - (V + V_t)^2 \quad (13)$$

respectively. Now, it should be noticed that the maximum value of  $V$  is greater than one

depending on the value of the parameter

$$e = \frac{\mu_0 \varepsilon}{\sin \alpha - \mu_0 \cos \alpha} \quad (14)$$

For instance, when  $e=0.1$  the maximum velocity is about 1.05. It has been assumed in deriving (12) and (13) that  $C_L=0$ . This assumption is justified, because the difference in maximum velocities attainable when  $C_L A=0.01m^2$  and  $C_L A=0$  is only 0.15%, *vide* equations (6)...(9).

The nondimensional, nonlinear differential equations (9), (10), (12) and (13) can be solved in closed form. The solutions of (9) and (10) are

$$T = \frac{1}{2} \ln \frac{(1-V_0-V_t)(1+V_t+V)}{(1+V_0+V_t)(1-V_0-V)} \quad (15)$$

and

$$S = \frac{1+V_t}{2} \ln \frac{1+V_0+V_t}{1+V_t+V} + \frac{1-V_t}{2} \ln \frac{1-V_0-V_t}{1-V_t-V} \quad (16)$$

Solutions of (12) and (14) are

$$T = \frac{1}{\sqrt{e^2 - 4eV_t + 4}} \ln \left( \frac{(\sqrt{e^2 - 4eV_t + 4} - e + 2V_t + 2V)(2V_0 - \sqrt{e^2 - 4eV_t + 4} - e + 2V_t)}{(\sqrt{e^2 - 4eV_t + 4} + e - 2V_t - 2V)(2V_0 + \sqrt{e^2 - 4eV_t + 4} - e + 2V_t)} \right) \quad (17)$$

and

$$S = \frac{(2V_t - e)}{2\sqrt{e^2 - 4eV_t + 4}} \ln\left(-\frac{(2V_0 + \sqrt{e^2 - 4eV_t + 4} - e + 2V_t)(\sqrt{e^2 - 4eV_t + 4} + e - 2V_t - 2V)}{(2V_0 - \sqrt{e^2 - 4eV_t + 4} - e + 2V_t)(\sqrt{e^2 - 4eV_t + 4} + e + 2V_t + 2V)}\right) + \frac{1}{2} \ln \frac{V_0^2 + V_t^2 + 2V_0V_t - eV_0 - 1}{V^2 + V_t^2 + 2VV_t - eV - 1}$$

(18)

In the derivation of the solutions, the symbolic mathematics manipulation program DERIVE has been employed. Also manual work is, of course, needed.

The solutions (15) and (16) reduce to the ones given in [1] when the corresponding simplifying assumptions are made. In the same way (17) and (18) give same results as (15) and (16) when  $e=0$ . By substituting the right-hand sides of (4)...(8) to the corresponding variables, we obtain explicit solutions in terms of the physical parameters. These equations are, however, too lengthy for printing.

The result (16) has been compared with the numerical solutions given in the figures of [2] by using the same values of the physical parameters. The results agree within drawing accuracy.

## EXAMPLES

In the two first examples we show the results (15) and (16), i.e. when coefficient of friction is independent of velocity.

Figure 1 shows the nondimensional velocity as a function of nondimensional time for two cases: initial velocity and velocity of wind equal to zero and both nonzero. The velocity

approaches asymptotically the maximum velocity.

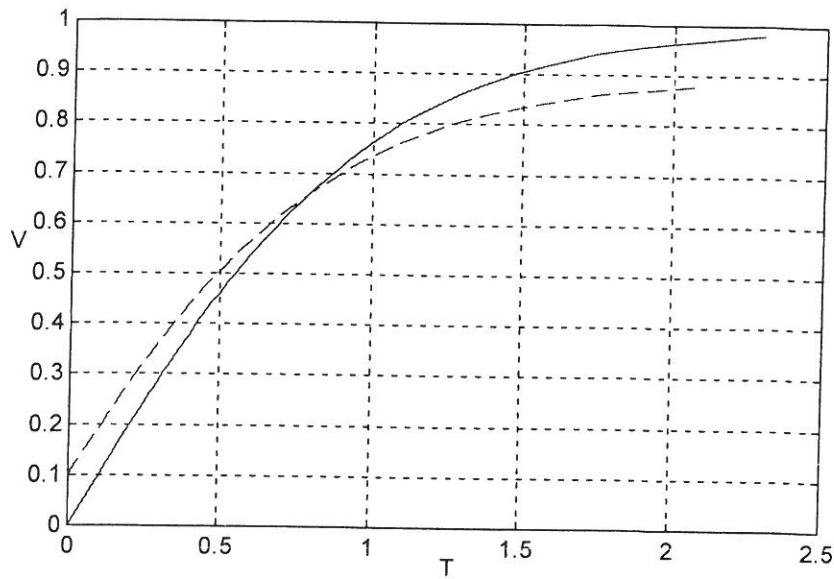


Figure 1. Nondimensional velocity as a function of nondimensional time. Solid line  $V_0=V_i=0$  and dashed line  $V_0=V_i=0.1$ , respectively.

Figure 2 represents nondimensional velocity as a function of nondimensional distance. Again the velocity approaches asymptotically the maximum velocity. It can also be seen that although the proportion of time is quite long before the gain of the initial velocity is “consumed” by the effect of wind (Fig.1), however, the corresponding portion of distance is rather short.

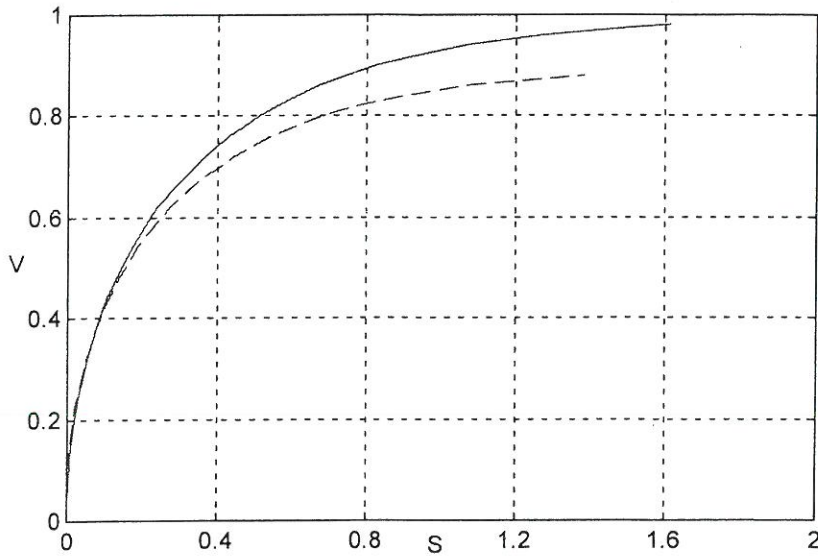


Figure 2. Nondimensional velocity as a function of nondimensional distance. Solid line  $V_0=V_t=0$  and dashed line  $V_0=V_t=0.1$ , respectively.

In the following two examples the assumption of velocity dependent friction is used. The parameter values in (14) are chosen to be  $\varepsilon=.5$ ,  $\mu_0=0.06$  and  $\alpha$  about  $20^\circ$ , which gives  $e=0.1$ .

Figures 3 and 4 show the nondimensional velocity as a function of the nondimensional time and the nondimensional distance, respectively.

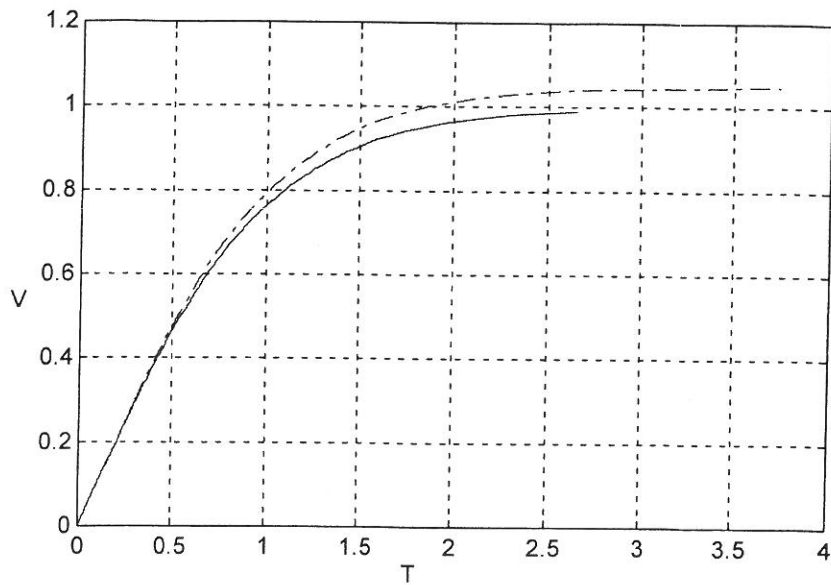


Figure 3. Nondimensional velocity as a function of nondimensional time. Solid line  $e=0$  and dashdot line  $e=0.1$ , respectively.

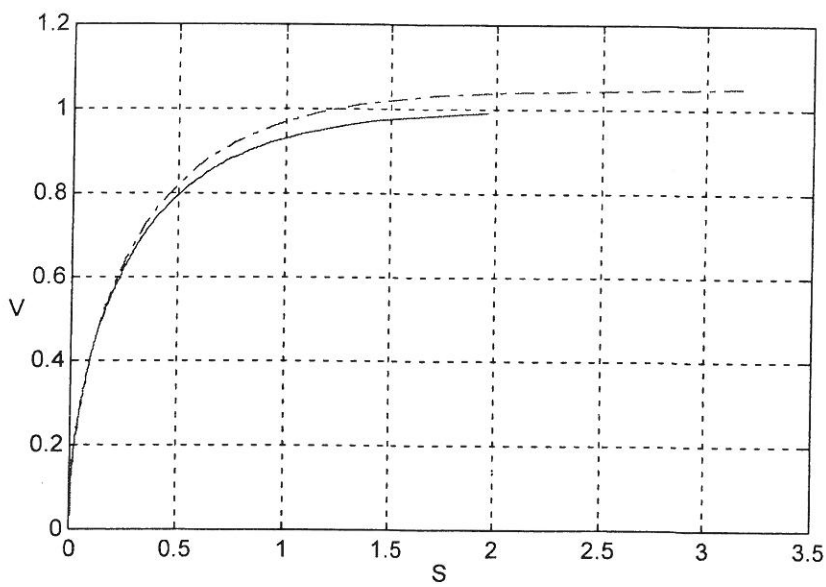


Figure 4. Nondimensional velocity as a function of nondimensional distance. Solid line  $e=0$  and dashdot line  $e=0.1$ , respectively.

## CONCLUDING REMARKS

A novel model for speed skiing is developed. The model takes into account gravity, velocity dependent friction and air resistance. Closed form solutions time versus velocity and distance versus velocity have been obtained. The results can be used to estimate the effect of various parameters. As a side result the conclusion of Ward-Smith and Clements that the effect of aerodynamic lift can be neglected has been confirmed, because it appears that with the upper bound estimate of the term  $C_L A$ , the maximum velocity is only 0.15% larger than when  $C_L A = 0$ .

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