MODELS OF SPRINTING BASED ON NEWTON'S SECOND LAW OF MOTION AND THEIR COMPARISON

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ABSTRACT

Four different sprint models, differing in the form of the resistive law, are presented and compared. An expression for the internal resistive force, in terms of physiological quantities, is derived. Analytical solutions for the distance-time relationship, when available, are given. The physiological parameters of the runner for each model are estimated by a least squares fitting procedure to Olympic 100 m data. Analysis of the residual error reveals that the linear-quadratic resistive law gives the best fit to the data.

INTRODUCTION

The first mathematical theories of running were inspired by Professor A. V. Hill and his co-workers (Furusawa *et al.*, 1927). Hill's model of sprinting was based on Newton's second law of motion – the rate of change of momentum equals the sum of the applied forces – to derive a relation between the running speed v, distance s and time from rest t. The model used two physiological parameters to characterize the sprinter, the maximum propulsive force per unit mass f and the parameter σ related to the internal losses associated with the runner's action. Furusawa *et al.* (1927) suggested that the resistive force should be modeled by a term linear in v, because the dominant resistive effects from running accrue from frictional losses within the body itself. Also Keller (1973, 1974) used this model in his theory of competitive running. Alexandrov and Lucht (1981) and many others have based their studies on Hill's sprinting model. The equation of motion for the sprinter is accordingly

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = f - \sigma v \,, \qquad (1)$$

where s is the distance measured from the start of the race, v the sprinter's instantaneous velocity and t time from rest. Another early assumption, suggested by the measurements of Margaria *et al.* (1963) and utilized in the theories of Ward-Smith (1983, 1985a, 1985b), was that the resistive force is independent of v .

Senator (1982) added the effects of air resistance to equation (1). The aerodynamic drag D can be expressed in the form

$$D = \frac{1}{2} C_D A \rho v^2 , \tag{2}$$

where ρ is the air density, A the frontal projected area of the runner and C_D the drag coefficient. Senator combined the internal resistive force F_R and the external air resistance D by the empirical law

$$F_R + D = c v^n , \tag{3}$$

where c and n are constants. However, according to Ward-Smith (1985a) there can be no physical justification for this arbitrary association of the terms.

Vaughan (1983a,b) used a modification of these approaches, writing the equation of motion in the form

$$\frac{dv}{dt} = A - B v^{\alpha} - C v^2 , \tag{4}$$

where A, B, C and α are constants. By measuring the performance of four university sprinters, Vaughan (1983a) obtained values for these parameters. The quantity α was evaluated as 0.7, whilst A ranged from 10.2 to 10.65 ms^{-2}, B from 1.92 to 2.02 $\mathrm{m}^{0.3}\mathrm{s}^{-1.3}$ and C from 0.0048 to 0.0053 m^{-1}.

In the next section we show that the internal resistive force may be written, tentatively, as $k_R v^2$. Combining with the external air resistance $k_D v^2$ yields the equation of motion

$$\frac{dv}{dt} = f - k v^2 , \tag{5}$$

where f is the propulsive force per unit mass and

$$k = k_R + k_D . \tag{6}$$

A closer examination reveals, however, that the coefficient k_D can be considered as a constant, whereas k_R can not. This stems from the fact that k_R contains, besides the runner's physiological parameters, also the stride lenght which varies during the initial acceleration. It can be inferred from the experimental data by Ae *et al.* (1992) that to a good approximation the product $k_R v \equiv k_r$ may be held as a constant. This leads in fact back to the linear model proposed by Furusawa *et al.* (1927) for the internal resistive force. Consequently, the equation of motion becomes now

$$\frac{dv}{dt} = f - k_r v - k_D v^2 , \tag{7}$$

where the internal loss cofficient k_r and the drag coefficient k_D are considered as constants.

We have now four different two-parameter models for sprinting: linear (L), Vaughan (Va), quadratic (Q) and linear-quadratic (LQ) models in equations (1), (4), (5) and (7), respectively. In the sequel we shall make a comparison between these theoretical models utilizing measured running data from the Olympic Games in Seoul 1988.

MODEL FOR PROPULSIVE AND INTERNAL RESISTIVE FORCES

We model the runner's leg by a straight, rigid body rotating about an axis through the hip joint. The torque on the leg supplied by the runner is M. The part M_{rot} is needed to rotate the leg and the part $M - M_{rot}$ pushes the runner forwards. The rotation equation of the leg is

$$I\frac{d^2\phi}{dt^2} = I\frac{d\omega}{dt} = M_{rot} , \tag{8}$$

where I is the moment of inertia of the leg about the rotational axis, ϕ the angle between the leg and vertical axis and ω the angular velocity. The solution to equation (8) subject to the initial conditions $\phi(0) = 0$, $\omega(0) = -\omega_0$ is

$$\phi = \frac{1}{2}\frac{M_{rot}}{I}t^2 - \omega_0 t . \tag{9}$$

The time for a half period of the leg is obtained from $\phi(t) = 0$ leading to

$$T_{1/2} = \frac{2I\omega_0}{M_{rot}} . \tag{10}$$

If ℓ is the stride lenght, we obtain for the runner's velocity

$$v = \frac{\ell}{T_{1/2}} = \frac{\ell}{2I\omega_0}M_{rot} . \tag{11}$$

On the other hand, the initial angular velocity is

$$\omega_0 = \frac{v}{d}, \tag{12}$$

where d is the length of the leg. Equations (11) and (12) lead to

$$M_{rot} = \frac{2I\omega_0}{\ell} v = \frac{2Iv^2}{d\ell}. \tag{13}$$

The force F which pushes the runner forward may now be written as

$$F = \frac{M - M_{rot}}{d} = \frac{M}{d} - \frac{2Iv^2}{d^2\ell} \tag{14}$$

and the equation of motion for the sprinter becomes

$$m\frac{dv}{dt} = \frac{M}{d} - \frac{2I}{d^2\ell}v^2 - \frac{1}{2}C_D A\rho v^2 = \frac{M}{d} - \left(\frac{2I}{d^2\ell} + \frac{1}{2}C_D A\rho\right)v^2 . \tag{15}$$

We write equation (15) in the form

$$\frac{dv}{dt} = f - kv^2 , \tag{16}$$

where

$$f = \frac{M}{md} \tag{17}$$

and

$$k = \frac{2I}{d^2\ell m} + \frac{1}{2}\frac{C_D A\rho}{m} \equiv k_R + k_D . \tag{18}$$

Note that $k_R \propto 1/\ell$, whereas

$$k_r \equiv k_R v = \frac{2I}{d^2 m}\frac{1}{T_{1/2}} \propto 1/T_{1/2} . \tag{19}$$

Measured data by Ae *et al.* (1992) shows that the stride lenght ℓ and stride period $T_{1/2}$ acquire their final values (plus small fluctuations) around 25 m and 10 m, respectively. Consequently, k_r may be considered as a constant to a better approximation than k_R. We see now that the internal resistive force per unit mass may be modeled by a term linear in v and supplemented by the air resistance leads to the equation of motion (7).

SOLUTIONS FOR THE SPRINTING MODELS

Linear Model

The well-known solution to equation (1) subject to the initial conditions $s(0) = 0$, $v(0) = v_0$ is (Alexandrov *et al.*, 1980)

$$v(t) = \frac{f}{\sigma}(1 - e^{-\sigma t}) + v_0 e^{-\sigma t}, \tag{20}$$

$$s(t) = \frac{f}{\sigma} t + \left(\frac{f}{\sigma^2} - \frac{v_0}{\sigma} \right)(e^{-\sigma t} - 1). \tag{21}$$

Vaughan Model

The solution to equation (4) subject to the initial conditions $s(0) = 0$, $v(0) = v_0$ can be found only numerically.

Quadratic Model

The solution $t = t(s)$ to equation (5) subject to the initial conditions $t(0) = 0$, $v(0) = v_0$ may be shown to be

$$t = \frac{1}{\sqrt{kf}} \ln \left\{ \frac{1}{1 + \sqrt{\frac{k}{f}} v_0} \left(e^{ks} + \sqrt{e^{2ks} - 1 + \frac{k}{f} v_0^2} \right) \right\}. \tag{22}$$

Equation (22) may be inverted for the relation $s = s(t)$.

Linear - Quadratic Model

Equation (7) belongs to the class of *Riccati* equations. Utilizing the transformation

$$v = \frac{1}{k_D} \frac{w'}{w} \tag{23}$$

one obtains the equivalent second order *linear* differential equation

$$\frac{d^2 w}{dt^2} + k_r \frac{dw}{dt} - k_D f w = 0. \tag{24}$$

The solution to equation (24) is

$$w(t) = C_1 e^{\zeta_1 t} + C_2 e^{\zeta_2 t} , \tag{25}$$

where the roots of the characteristic equation are

$$\zeta_{1,2} = \frac{-k_r \pm \sqrt{k_r^2 + 4k_D f}}{2} . \tag{26}$$

Utilizing (25) in (23) and defining a new constant $C = C_2 / C_1 \quad (C_1 \neq 0)$ gives

$$v(t) = \frac{1}{k_D} \frac{\zeta_1 e^{\zeta_1 t} + C \zeta_2 e^{\zeta_2 t}}{e^{\zeta_1 t} + C e^{\zeta_2 t}} . \tag{27}$$

Integration of (27) with the initial conditions $s(0) = 0$, $v(0) = v_0$ results in

$$s(t) = \frac{1}{k_D} \ln \left| \frac{e^{\zeta_1 t} + C e^{\zeta_2 t}}{1 + C} \right| , \tag{28}$$

where

$$C = - \frac{\zeta_1 - v_0 k_D}{\zeta_2 - v_0 k_D} . \tag{29}$$

ESTIMATION OF THE SYSTEM PARAMETERS

Since the solution to the Vaughan model cannot be found analytically, a numerical gradient method was used to fit the differential equations (1), (4), (5) and (7) into the measured data of the Olympic Games in Seoul 1988. Note that all the equations of motion can be written in the form

$$\ddot{s}(t) + B \dot{s}(t)^{\alpha} + C[\dot{s}(t) - w] \left| \dot{s}(t) - w \right| = A , \tag{30}$$

where the contribution of the wind velocity w has been added. We take the inverse of equation (30) and fit it into the data in the sense of least squares. After some manipulation the inverse equation takes the form

$$t''(s) - C\,[1 - t'(s) w] \left| \frac{1}{t'(s)} - w \right| t'(s)^2 - B t'(s)^{3-\alpha} + A t'(s)^3 = 0 . \tag{31}$$

Since the wind velocity in men's 100 m final was +1.10 ms^{-1}, the wind corrected data is calculated first. We fit equation (31) with $w = 0$ into the data to get a first approximation for the parameters A and B. Then we estimate the effect of the wind on the running times at the intermediate stations and repeat the procedure until convergence for the values of A and B is reached. The data with and without wind is presented in Table 1 for Ben Johnson (BJ), Carl Lewis (CL) and Linford Christie (LC), where the corrected data is the average over the four models .

TABLE 1. *Original (t_{or}) and wind corrected (t) data (in seconds) for Ben Johnson, Carl Lewis and Linford Christie in men's 100 m final in Seoul 1988.*

	BJ		CL		LC	
s(m)	t_{or}	t	t_{or}	t	t_{or}	t
10	1.83	1.83	1.89	1.89	1.92	1.92
20	2.87	2.88	2.96	2.97	2.97	2.98
30	3.80	3.81	3.90	3.91	3.92	3.94
40	4.66	4.68	4.79	4.81	4.81	4.83
50	5.50	5.53	5.65	5.68	5.66	5.69
60	6.33	6.36	6.48	6.52	6.50	6.54
70	7.17	7.21	7.33	7.37	7.36	7.40
80	8.02	8.07	8.18	8.23	8.22	8.27
90	8.89	8.94	9.04	9.10	9.09	9.15
100	**9.79**	**9.85**	**9.92**	**9.98**	**9.97**	**10.00**

The variation in the wind corrected data from model to model was very small, typically ±0.01 s in the range 50 - 100 m. However, in estimating the parameters for each model, we have used wind correction by the very same model.

TABLE 2. *Parameters evaluated from the wind corrected data.*

	Linear		Vaughan		Quadratic		Linear-Quadratic	
	A	B	A	B	A	$B+C$	A	B
	($\mathrm{ms^{-2}}$)	($\mathrm{s^{-1}}$)	($\mathrm{ms^{-2}}$)	($\mathrm{m^{0.3}s^{-1.3}}$)	($\mathrm{ms^{-2}}$)	($\mathrm{m^{-1}}$)	($\mathrm{ms^{-2}}$)	($\mathrm{s^{-1}}$)
BJ	9.00	0.757	10.85	1.84	6.59	0.0485	8.82	0.711
CL	8.15	0.688	9.78	1.66	6.01	0.0448	7.98	0.642
LC	8.09	0.687	9.71	1.65	5.96	0.0448	7.92	0.641

The results of the parameter estimations are presented in Table 2 and the calculated final times with +1.10 $\mathrm{ms^{-1}}$ wind and quadratic residual errors for each model in Table 3. The constant $C = \frac{1}{2} C_D A \rho$ was set to the value $C = 0.0033\ \mathrm{m^{-1}}$ in all calculations. The quadratic residual error is the sum of the squared time errors over the ten time stations.

TABLE 3. *Calculated final times (t_c) and quadratic residual errors (ε) (in seconds and seconds^2, respectively).*

		Linear		Vaughan		Quadratic		Linear-Quadratic	
	t_{or}	t_c	$\varepsilon \cdot 10^3$	t_c	$\varepsilon \cdot 10^3$	t_c	$\varepsilon \cdot 10^3$	t_c	$\varepsilon \cdot 10^3$
BJ	9.79	9.74	7.56	9.73	9.19	9.75	8.32	9.74	7.26
CL	9.92	9.89	2.98	9.88	3.55	9.91	8.03	9.89	2.96
LC	9.97	9.94	3.60	9.93	5.25	9.96	4.99	9.94	3.28
$\varepsilon_{tot} \cdot 10^3$			14.14		17.99		21.34		13.50

DISCUSSION

It can be seen from the residual errors of Table 3, that the LQ-model gives the best fit to the data, then come the L-, Va- and Q-models in order of decreasing fit. It can also be seen, that BJ fits to all the models much poorer than the two other runners. The evident explanation lies in the well known fact that BJ's speed attains its maximum value relatively early around 50-70 m (Van Coppenolle *et al.*, 1989) and then decreases, whereas all the models predict an increasing velocity profile which asymptotically levels off towards a constant value. The average residual error per station and runner for the LQ-model is about 0.01 s, which means a fairly good fit throughout the run. The success of the LQ-model is by no means surprising as was already reasoned in connection with equation (19).

It is of interest to compare the values of the calculated physiological parameters of Table 2 with the corresponding values in existing literature. The values of A and B for the L- and LQ-models in Table 2 are relatively close to each other, although the values of the L-model seem to lie systematically somewhat above those of the LQ-model. Furusawa *et al.* (1927) have reported for the L-model the values $A = 8.6\ \mathrm{ms}^{-2}$ and $B = 1.0\ \mathrm{s}^{-1}$ (mean value), Keller (1973) gives $A = 12.2\ \mathrm{ms}^{-2}$ and $B = 1.12\ \mathrm{s}^{-1}$, Vaughan and Matravers (1977) have reported $A = 10.26\ \mathrm{ms}^{-2}$ and $B = 0.963\ \mathrm{s}^{-1}$, whereas Woodside (1991) ends up with $A = 14.4\ \mathrm{ms}^{-2}$ and $B = 1.35\ \mathrm{s}^{-1}$. It is quite evident that the A-values by Woodside and Keller are too high since recent starting block force data for international top level sprinters (Van Coppenolle *et al.*, 1989) gives initial acceleration values in the range of 12.5 – 12.9 ms^{-2} for the push off from the blocks. This must set an upper limit for the coefficient A, since the vigorous two-leg push in a leaning position with continuous ground contact evokes a strong propulsive reaction. The A-value by Vaughan and Matravers is determined as the average of the initial slope of ten velocity-time recordings. Consequently, their value is strongly affected by the push off from the blocks leading to an evident overestimation as well.

Our values for the Q-model, typically $A \simeq 6.0\ \mathrm{ms}^{-2}$ and $B + C \simeq 0.045\ \mathrm{m}^{-1}$, are much lower than the corresponding values $10.3\ \mathrm{ms}^{-2}$ and $0.095\ \mathrm{m}^{-1}$ reported by Vaughan and Matravers (1977). It should be noted that the data of Vaughan and Matravers is collected only at three points; the start, 27.5 m and the location of maximum velocity, whereas the data used by us is measured at ten stations with 10 m intervals. It is quite evident that the start weighted data of Vaughan and Matravers leads to larger values of the parameter A.

Vaughan (1983a) has stated that the optimal value for the exponent α in equation (4) is $\alpha = 0.7$. With this exponent he has got for A and B typically the values $10.5\ \mathrm{ms}^{-2}$ and $2.0\ \mathrm{m}^{0.3}\mathrm{s}^{-1.3}$. These are quite close to our results for BJ. However, one should note that BJ's propulsive force in Seoul certainly was unique. Consequently, the A-value for BJ should be clearly larger than for the university sprinters used by Vaughan. The discrepancy can once again be explained by the start weighted data used by Vaughan in his parameter evaluations.

Vaughan and Matravers (1977) have also studied the square root, hyperbolic and exponential models. They found that the square root model gives the best fit. However, later Vaughan (1983a) found that his $\alpha = 0.7$ exponent law gives a still better fit to the data. As far as we know, the LQ-model has not been used earlier.

CONCLUSIONS

Our results show that the linear-quadratic resistance law fits best to the real Olympic 100 m data. It is interesting to note that the linear law was already suggested by Furusawa, Hill and Parkinson (1927). There is, however, a noteworthy difference between the interpretation of the origin of the linear term in Hill's theory and in the LQ-model presented in this paper: Hill and colleagues invoked the concept of the viscosity of the muscles, while we ended up to the linear term by writing the rotational equation of the leg combined to the observed fact that the stride frequency can, to a good approximation, be considered as constant. It must be noted that already Fenn (1930) strongly criticized the viscosity concept and proposed that the resistive force could be the result of tension of antagonistic muscles and other kinesiological factors. In view of our derivation of the LQ-model (see equations (8)-(19)), it is evident that the resistive force stems simply from the rotational inertia of the leg, whereas the energy losses occur in the antagonistic muscles during the decelerating phases of the back and forth motion of the legs.

In a future work the comparison of the models should be performed for a more comprehensive data including runners at various levels. Also, the model should be generalized to account for the observed decrease in speed at the last stage of the run.

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