STRESS ENERGY AND PSEUDO STRESS ENERGY IN THE ANALYSIS OF A SUSPENSION BRIDGE WITH INCLINED HANGERS

Pertti Holopainen

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ABSTRACT

The compability equations for a suspension bridge with inclined hangers have been derived. The formulation of the principle of stationary complementary work including the change of temperature derived by the writer has been applied.

INTRODUCTION

The principle of stationary complementary work for physically non-linear but geometrically linear structures has been presented by Engesser (1889) /1/. For a long time one has belived that a theoretical analysis of geometrically non-linear continuum is not possible as a function of stresses only. Not until in 1970 L.M. Zubov /2/ published the principle of stationary complementary work for a non-linear continuum as a function of the Piola stress tensor only. A little earlier (1967) Oran /3/ proposed the complementary energy consept for geometrically non-linear structures. An other formulation has been presented by the writer (1974) /4/, /5/. This formulation has been applied in this paper in the analysis of a suspension bridge with inclined hangers. To analyse the suspension bridge with inclined hangers ortogonal to the suspension cable has been suggested by professor Paavola. A treatise on this subject has been made by Holopainen and Mikkola (1972) /6/. In this paper the same subject has been considered by a somewhat different formulation and by taking the change of temperature and the shear deformations of the stiffening beam into account.

COMPLEMENTARY WORK

The complementary work has been derived by the writer in 1974 /4/

$$W_c = \Sigma H_{ij}(\overline{\Delta h}_{ij} + \Delta h^*_{ij}) + U_c \quad (1)$$

for geometrically nonlinear elastic structures. In (1)

W_c	external complementary work
U_c	stress energy
H_{ij}	element node force at the node i and in the fixed direction j
$\overline{\Delta h}_{ij}$	relative node displasement caused by the elements rigid body rotation only
Δh^*_{ij}	relative node displacement caused by the change of temperature
$\Sigma H_{ij} \overline{\Delta h}_{ij}$	the pseudo stress energy. The name has been given /5/ by the writer.

If the rigid body rotations of all elements are infinite as it can be assumed in geometrically linear structures, the pseudo stress energy approaches to zero and it can be set

$$W_c = U_c$$

where W_c is the same as the Engessers work /1/.

<u>Beam element</u>. Consider a beam element as shown in Fig. 1.

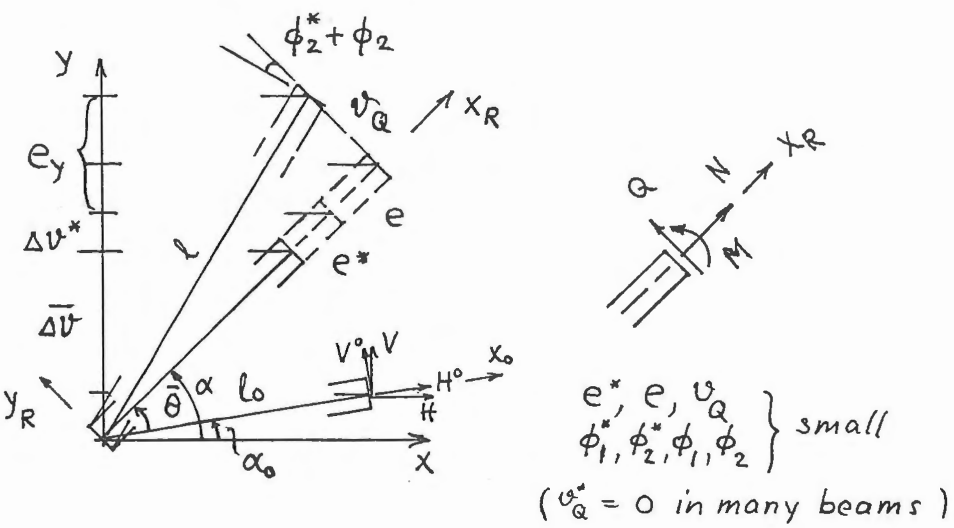

Figure 1. Timoshenko beam element.

The external complementary work of element

$$W_c = H(\overline{\Delta u} + \Delta u^*) + V(\overline{\Delta v} + \Delta v^*) + M_2(\bar{\theta} + \phi_2^*) +$$

$$M_1(-\bar{\theta} + \phi_1^*) + U_c(N,Q,M) \tag{2}$$

where H, V, M_1, M_2 are elements node force in the fixed clobal co-ordinate frame and N,Q,M element forces in the rigid body co-ordinate frame. The stress energy $U_c(N,Q,M)$ has been written so as for geometrically linear (or slightly nonlinear) elements in general, with well known formulas. For a physically linear beam element

$$U_c = \frac{1}{2} \left[\int_0^{l_0} \frac{N^2}{EA} ds + \zeta \int_0^{l_0} \frac{Q^2}{GA} ds + \int_0^{l_0} \frac{M^2}{EI} ds \right]. \tag{3}$$

But here the N, Q and M are considered as functions of H, V, M_1 and M_2. So it can be obtained

$$\frac{\partial W_c}{\partial H} = \overline{\Delta u} + \Delta u^* + \frac{\partial U_c}{\partial N}\frac{\partial N}{\partial H} + \frac{\partial U_c}{\partial Q}\frac{\partial Q}{\partial H} = \overline{\Delta u} + \Delta u^* + e\cos\alpha - v_Q \sin\alpha$$

$$\frac{\partial W_c}{\partial V} = \overline{\Delta v} + \Delta v^* + \frac{\partial U_c}{\partial N}\frac{\partial N}{\partial V} + \frac{\partial U_c}{\partial Q}\frac{\partial Q}{\partial V} = \overline{\Delta v} + \Delta v^* + e\sin\alpha + v_Q \cos\alpha$$

$$\frac{\partial W_c}{\partial M_1} = -\overline{\theta} + \phi_1^* + \frac{\partial U_c}{\partial Q}\frac{\partial Q}{\partial M_1} + \frac{\partial U_c}{\partial M}\frac{\partial M}{\partial M_1} = -\overline{\theta} + \phi_1^* + v_Q\frac{1}{l_0} + \phi_1 \qquad (4)$$

$$\frac{\partial W_c}{\partial M_2} = \overline{\theta} + \phi_2^* + \frac{\partial U_c}{\partial Q}\frac{\partial Q}{\partial M_2} + \frac{\partial U_c}{\partial M}\frac{\partial M}{\partial M_1} = \overline{\theta} + \phi_2^* - v_Q\frac{1}{l_0} + \phi_2$$

when

$$\left.\begin{aligned} N &= H\cos\alpha + V\sin\alpha \\ Q &= -H\sin\alpha + V\cos\alpha = \frac{1}{l_0}(M_1 - M_2) \\ M &= M_1 + \frac{x}{l_0}(M_2 - M_1) \end{aligned}\right\} \qquad (5)$$

The relative displacements

$$\begin{aligned} \overline{\Delta u} &= l_0(\cos\alpha - \cos\alpha_0) \\ \overline{\Delta v} &= l_0(\sin\alpha - \sin\alpha_0) \\ \Delta u^* &= \alpha_t \Delta T\, l_0 \cos\alpha \\ \Delta v^* &= \alpha_t \Delta T\, l_0 \sin\alpha \end{aligned} \qquad (6)$$

where

$$\alpha = \alpha_0 + \overline{\theta}$$

and $\bar{\theta}$ is the rigid body rotation of the beam element. The rotation can be calculated as a function of forces

$$\bar{\theta} = \begin{cases} \psi - \bar{\phi} & , \text{ when the beam is tensioned} \\ \psi + \bar{\phi} - \pi \, \mathrm{sgn}(\psi) & , \text{ when the beam is compressed} \end{cases} , \tag{7}$$

where

$$\psi = \arc\tan\left(\frac{V^0}{H^0}\right)$$

$$\bar{\phi} = \overline{\arc\sin}\left(\frac{-\Delta M}{lF}\right)$$

and (8)

$$\Delta M = M_2 - M_1$$

$$F = \sqrt{H^{0\,2} + V^{0\,2}} = \sqrt{H^2 + V^2}$$

and H^0, V^0 are the components of F referred to the fixed local co-ordinate frame $0\,x_0\,y_0$ attached to the element in its initial position as shown in Fig. 1.

When the deformations e, $\underline{v}_Q$ and κ are small, it can be substituted by $l = l_0$ in $(8)_2$. Otherwise $\bar{\theta}$ can be calculated by an iteration.

It can be imagined that when the assembly of structural elements are fixed in the rigid body rotation state (obtained as a function of unknown forces), the compatibity can be reached by pure deformations only.

It can be noted that the formula of W_c in (2) includes the angle $\bar{\theta}$ both explicitly and implicitly and $\bar{\theta}$ is a function of forces. By the derivation of the equations (4) the term including the derivatives of $\bar{\theta}$ has been dropped for the reason that the beam element is in the moment equilibrium in the final state (with unknown momentarms). This has been pointed in detail by the writer in /8/.

When the beam element is pin-ended, the end moments M_1, M_2 and hence $\bar{\phi}$ are equal to zero. The equations (2...8) are still applicable for the pin-jointed bars, so as for cable or hanger elements.

When eliminating the element node forces one must note that in plane two force-equilibrium equations are usable only, when the element rotation or deformations are large /9/. Otherwise the moment equilibrium equation is usable, too. In the analysis of gemetrically non-linear structures containing beam elements and pin-endet bar elements only, the number of unknown forces

$$n = 6E + 4E^0 - 2(E+E^0) - 3S - 2S^0 - k \tag{9}$$

where E is the number of beam elements, E^0 is the number of pin-ended bar elements, S is the number of moment nodes, S^0 the number of force nodes and k is the number of specified node forces equal to zero. For example in the suspension bridge considered later $M_1 = M_6 = 0$, $E = 5$, $E^0 = 9$, $S = 4$, $S^0 = 4$, $k = 2$ and substituting in eq. (9) one can calculate $n = 16$. The releases and unknown forces have been shown in Fig. 2b and c. When the same structure is geometrically linear, the number of unknown forces

$$n = 6E + 4E^0 - 3(E+E^0) - 3S - 2S^0 - k = 2 \tag{10}$$

e.g. X_1 and X_1' and the releases at the left end as shown in Fig. 2b and c.

SUSPENSION BRIDGE

Consider a suspension bridge as shown in Fig. 2a. If the towers are inclined, in the case (a) the anchor forces are larger and in the case (b) smaller than with the vertical towers. If the angles α_1 and α differ much from each others, the cable connection to the top of tower can be problematic except to the pedestrian bridges.

The structure containing the side cable and the tower on the foundation is sufficiently stiff. It has been replaced in this paper for the sake of simplicity by a horizontal and vertical springs, the stiffness of which corresponds to the one of supporting structure (Fig. 2b).

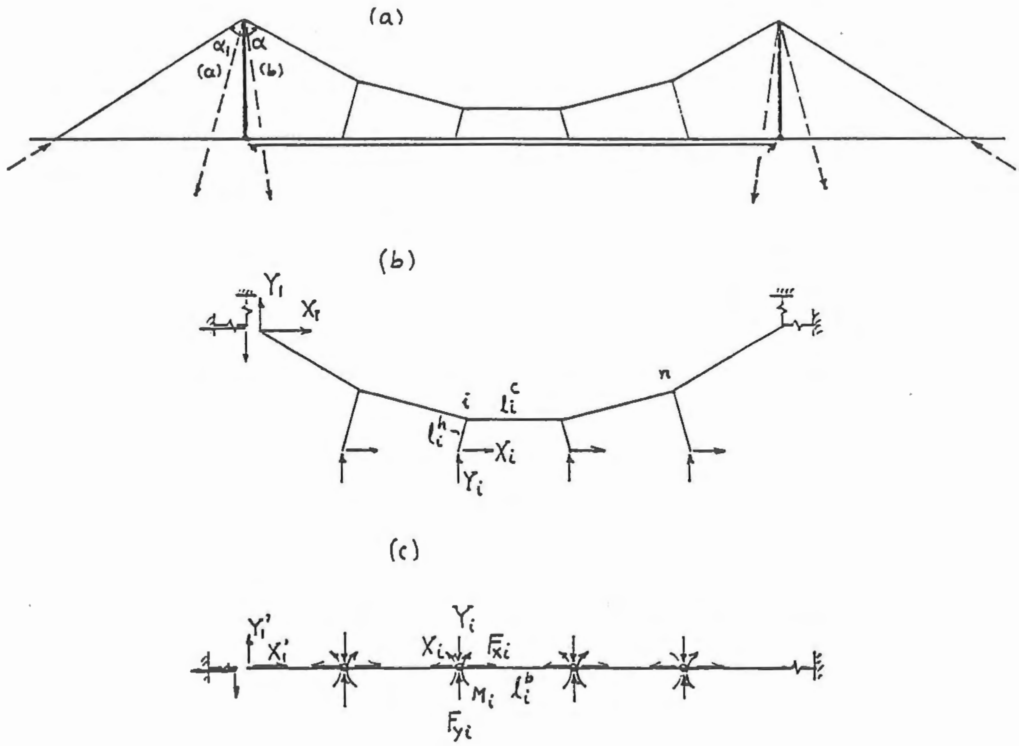

Figure 2. Suspension bridge and base structue.

The analysis is based on the following assumptions:

1. The stress-strain characteristics of the cable, hangers and the siffening beam are elastic.

2. The cable is perfectly flexible.

3. The own weights of the cable and hangers are neglected.

4. The geometry of the stucture and the internal forces are known at dead load.

As to the notatin, the subscripts d and l are used for dead load and live load, respectively. Thus, e.g. $H_i = H_{id} + H_{il}$ where H_{id} is the component due to dead load only and H_{il} is the change caused by live load. The superscripts c, h and b are used for cable, hanger and stiffening beam, respectively.

The base structure and the statically indeterminate forces X_i, Y_i and M_k are chosen as shown in Figs. 2b and c. The base structure has been divided into elements: the cable segments between hanger connection points, the hangers and the beam segments between hanger connection points.

The complementary energy

$$W_c = \sum_{i=1}^{n} \left[H_i^c \left(\overline{\Delta u_i^c} + \Delta u_i^{*v} \right) + V_i^c \left(\overline{\Delta v_i^c} + \Delta v_i^{*c} \right) + U_{ci}^c \right]$$

$$+ \sum_{i=2}^{n} \left[X_i \left(\overline{\Delta u_i^h} + \Delta u_i^{*h} \right) + Y_i \left(\overline{\Delta v_i^h} + \Delta v_i^{*h} \right) + U_{ci}^h \right]$$

$$+ \sum_{i=1}^{n} \left[H_i^b \left(\overline{\Delta u_i^b} + \Delta u_i^{*b} \right) + V_i^b \left(\overline{\Delta v_i^b} + \Delta v_i^{*c} \right) + U_{ci}^b \right]$$

$$+ \sum_{i=1}^{n} \left[M_i \left(-\bar{\theta}_i + \phi_i^* \right) + M_{i+1} \left(\bar{\phi}_i + \phi_{i+1}^* \right) \right] - W_{cd} \qquad (11)$$

where $M_1 = M_{n+1} = 0$

and the node forces of cable and beam elements

$$\left. \begin{aligned} H_i &= H_{id}^c - \sum_{k=1}^{i} X_{kl} \\ V_i^c &= V_{id}^c - \sum_{k=1}^{i} Y_{kl} \end{aligned} \right\} \quad (i = 1 \ldots n) \qquad (12)$$

$$\left. \begin{aligned} X_i &= X_{id} + X_{il} \\ Y_i &= Y_{id} + Y_{il} \\ M_i &= M_{id} + M_{il} \end{aligned} \right\} \qquad (13)$$

$$\begin{aligned} H_i^b &= H_{id}^b = -X'_{il} + \sum_{k=2}^{i} \left(X_{kl} - F_{xk} \right) \\ V_i^b &= V_{id}^b = -Y'_{il} + \sum_{k=2}^{i} \left(Y_{kl} - F_{yk} \right) \quad (i = 1 \ldots n) \end{aligned} \qquad (14)$$

Based on the principle of stationary complementary work derived in /4/ in this form it have to set

$$\frac{\partial W_c}{\partial X_{11}} = \sum_{i=1}^{n} \left[- \left(\overline{\Delta u}_i^c + \Delta u_i^{*c} \right) + e_{xi}^c \right] + e_{xo}^c + e_{x,n+1}^c = 0$$

$$\frac{\partial Y_c}{\partial Y_{11}} = \sum_{i=1}^{n} \left[- \left(\overline{\Delta v}_i^c + \Delta v_i^{*c} \right) + e_{yi}^c \right] + e_{yo}^c + e_{y,n+1}^c = 0 \qquad (15)$$

$$\frac{\partial X_c}{\partial X_{i1}} = \sum_{k=i}^{n} \left[- \left(\overline{\Delta u}_k^c + \Delta u_k^{*c} \right) + e_{xk}^c + \left(\overline{\Delta u}_k^b + \Delta u_k^{*b} \right) + e_{xk}^b \right] +$$

$$+ \left(\overline{\Delta u}_i^h + \Delta u_i^{*h} \right) + e_{xi}^h + e_{x,n+1}^c + e_{x,n+1}^b = 0$$

$$(i = 2...n)$$

$$\frac{\partial W_c}{\partial Y_{i1}} = \sum_{k=i}^{n} \left[- \left(\overline{\Delta v}_k^c + \Delta v_k^{*c} \right) + e_{yk}^c + \left(\overline{\Delta v}_k^b + \Delta v_k^{*b} \right) + e_{yk}^b \right]$$

$$\left(\overline{\Delta v}_i^h + \Delta v_i^{*h} \right) + e_{yi}^h + e_{y,n+1}^c = 0 \qquad (16)$$

$$(i = 2...n)$$

$$\frac{\partial W_c}{\partial X'_{11}} = \sum_{i=1}^{n} \left[- \left(\overline{\Delta u}_i^b + \Delta u_i^{*b} \right) + e_{ix}^b \right] + e_{x_0}^b + e_{x,n+1}^b = 0$$

$$\frac{\partial W_c}{\partial Y'_{11}} = \sum_{i=1}^{n} \left[\left(\overline{\Delta v}_i^b + \Delta v_i^{*b} \right) + e_{iy}^b \right] = 0 \qquad (17)$$

$$(i = 2...n)$$

$$\frac{\partial M_c}{\partial M_{i1}} = -\overline{\theta}_i + \overline{\theta}_{i-1} + \phi_{i1} + \phi_{i-1,2} + \phi_{i1}^* + \phi_{i-1,2}^* +$$

$$+ v_{Q_i} \frac{1}{l_{io}} - v_{Q_{i-1}} \frac{1}{l_{i-1,o}} = 0 \qquad (18)$$

$$(i = 2...n)$$

Professor Paavola /10/ has derived the three moment equations for stiffening beam of suspension bridge with vertical hangers.

In the equations (15)

$$\overline{\Delta u_i^c} = l_{io}^c \left(\cos\alpha_i^c - \cos\alpha_{io}^c \right)$$

$$\overline{\Delta v_i^c} = l_{io}^c \left(\sin\alpha_i^c - \sin\alpha_{io}^c \right) \tag{19}$$

$$\Delta u_i^* = l_{io}^c \, \alpha_t^c \, \Delta T \cos\alpha_i^c$$

$$\Delta v_i^* = l_{io}^c \, \alpha_t^c \, \Delta T \sin\alpha_i^c$$

and

$$\alpha_i^c = \alpha_{io}^c + \theta_i^c \tag{20}$$

Substituting the equations (16) and (17) instead of superscript c are the superscripts h (hangers) and b (beam) in the equations (19) and (20), respectively. In the equations (15)

$$e_{xi}^c = e_i^c(N_i^c) \cos\alpha_i^c$$

$$e_{yi}^c = e_i^c(N_i^c) \sin\alpha_i^c \tag{21}$$

and substituting in the equations (16) instead of c is the superscript h. In the equations (16) and (17)

$$e_{xi}^b = e_i^b(N_i^b) \cos\alpha_i^b - v_{i_Q} \sin\alpha_i^b$$

$$e_{yi}^b = e_i^b(N_i^b) \sin\alpha_i^b + v_{i_Q} \cos\alpha_i^b \tag{22}$$

where

$$-v_{i_Q} \sin\alpha_i^b \approx 0 \,. \tag{23}$$

The elements rigid body rotations (from the d-state i.e. from the Ox_{io} - axes)

$$\bar{\theta}_i^c = \psi_i^c - \arctan\left(\frac{V_i^{oc}}{H_i^{oc}}\right)$$

$$\bar{\theta}_i^h = \psi_i^h \quad , \qquad (24)$$

when the cable is perfectly flexible and the hangers pin-ended. The beam elements rigid body rotation

$$\bar{\theta}_i^b = \begin{cases} \psi_i^b - \bar{\phi}_i^b & \text{when the beam is tensioned} \\ \psi_i^b + \bar{\phi}_i^b - \pi \operatorname{sgn}(\psi_i^b) & \text{when the beam is compressed} \end{cases} \qquad (25)$$

The tension of the cable

A typical load deflection curve of a wire rope is shown in Fig 3.

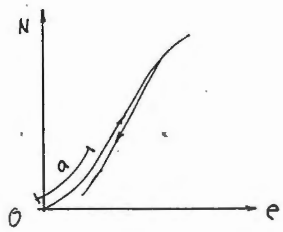

Figure 3.

The part a is due to internal geometric non-linearity of the wire rope. This phenomenon is more slight in locked-coil wire rope.

SUMMARY

The stationary principle of complementary work, where the complementary energy has been considered as a sum of stress energy and pseudo stress energy, is also suitable for the derivation of the compatibility equations for more complicated skeletal structures,when the base structure has been determined.

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Pertti Holopainen, Associate Professor, Tampere University of Technology, Tampere