ON THE CREEP AND STRESS RELAXATION OF CONCRETE

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1. Introduction

More than a thousand papers have been concerned with the description of experiments on the creep behaviour of concrete. Comparatively few papers relate to stress relaxation. Consequently, it seems very desirable that an expression be formulated to allow of prediction of the time dependence of stress relaxation upon the basis of creep values determined experimentally. Several tries have been made to link creep functions with relaxation functions. In most cases, this has been done by application of the methods developed in the theory of linear viscoelasticity. We will thus introduce first of all the basic equations of linear viscoelasticity.

2. Basic equations of linear viscoelasticity

Most equations of the theory of linear viscoelasticity are describable by means of rheological models. In the simplest case, rheological models can be composed of elastic elements that obey Hooke's law ($\sigma = E\cdot\varepsilon$), and viscous elements that obey Newton's law of viscosity ($\sigma = 3\eta\dot{\varepsilon}$).

The elastic element can be represented by a spring, and the viscous element by a dash pot.

If a viscous body and an elastic body are put in series, the result is the so-called Maxwell-body. In this case, the deformations of two elements are added (see figure 1):

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \quad \text{or} \quad \dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2$$

The following differential equation can then be written

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{3\eta} \tag{1}$$

From equation (1), the creep function is easily obtainable with $d\sigma/dt = 0$

$$\varepsilon = \frac{\sigma}{3\eta}\, t \tag{2}$$

By analogy, the relaxation function can be found with $d\varepsilon/dt = 0$:

$$\frac{\sigma}{\sigma_o} = e^{-\frac{E}{3\eta}t} \tag{3}$$

If the two elements are considered to be parallel, one usually speaks of a Voigt body. In this case, the stresses of each element are to be added:

$$\sigma = \sigma_1 + \sigma_2$$

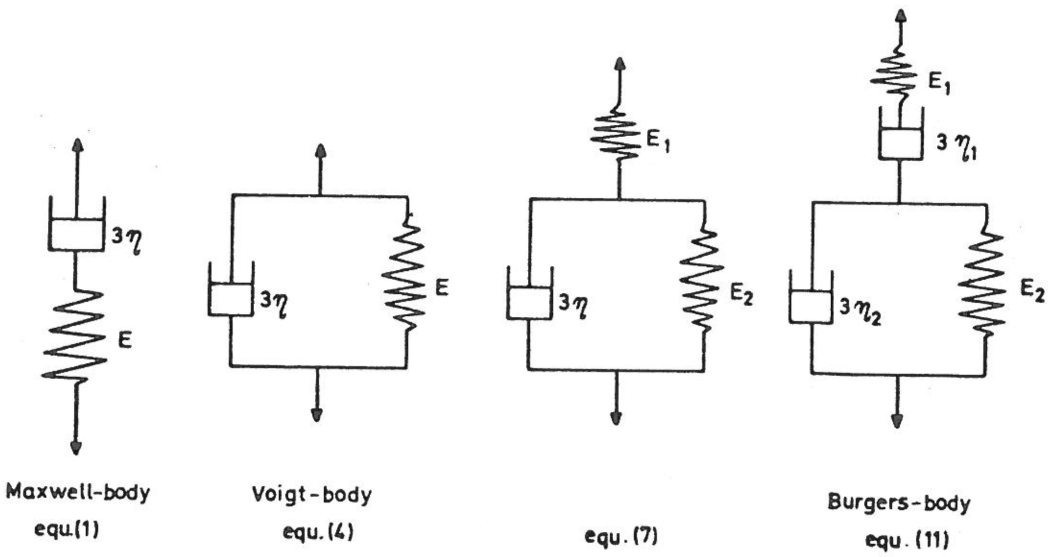

Fig. 1. Rheological models described by equations 1, 4, 7 and 10.

This leads to the differential equation:

$$\sigma = E\varepsilon + 3\eta \frac{d\varepsilon}{dt} \qquad (4)$$

The resulting creep function is $(d\sigma/dt = 0)$:

$$\varepsilon = \frac{\sigma}{E} (1 - e^{-\frac{E}{3\eta} t}) \qquad (5)$$

Consequently, the resulting relaxation function is $(d\varepsilon/dt = 0)$:

$$\sigma = E \cdot \varepsilon \quad \text{or} \quad \frac{\sigma}{\sigma_o} = 1 \qquad (6)$$

This means that a Voigt-body does not release stress under constant strain.

A third example will be examined. We add an elastic element to a Voigt-body. In this case, the deformations of the elastic element and of the Voigt-body have to be added:

$$\varepsilon_1 = \frac{\sigma}{E_1} \quad \text{and} \quad \sigma = E_2\varepsilon_2 + 3\eta\dot{\varepsilon}_2$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

The following differential equation can then be written:

$$\left(\frac{1}{E_1} + \frac{1}{E_2} \right)\sigma + \frac{3\eta}{E_1 \cdot E_2} \frac{d\sigma}{dt} = \varepsilon + \frac{3\eta}{E_2} \frac{d\varepsilon}{dt} \qquad (7)$$

or, in a more general way:

$$a_o\sigma + a_1 \frac{d\sigma}{dt} = b_o\varepsilon + b_1 \frac{d\varepsilon}{dt} \qquad (7a)$$

From equation (7), there are deducible the following creep function:

$$\varepsilon = \frac{\sigma}{E} \left[1 + \frac{E_1}{E_2} (1 - e^{-\frac{E_2}{3\eta} t}) \right] \qquad (8)$$

and the corresponding relaxation function:

$$\frac{\sigma}{\sigma_o} = 1 - \frac{E_1}{E_1 + E_2} \left(1 - e^{-\frac{E_1+E_2}{3\eta} t}\right) \tag{9}$$

Several authors have applied a slightly more complicated rheological model to describe the behaviour of concrete. In this Model, a Maxwell-body and a Voigt-body are put in series. This model is usually termed a Burgers-body. The differential equation for a Burgers-body is:

$$\sigma + \left(\frac{3\eta_1}{E_1} + \frac{3\eta_1}{E_2} + \frac{3\eta_2}{E_2} \right) \frac{d\sigma}{dt} + \frac{3\eta_2 \cdot 3\eta_1}{E_2 \cdot E_1} \frac{d^2\sigma}{dt} = \tag{10}$$

$$= 3\eta_1 \left(\frac{d\varepsilon}{dt} + \frac{3\eta_2}{E_2} \frac{d^2 t}{dt^2} \right)$$

which is the same as:

$$a_o\sigma + a_1 \frac{d\sigma}{dt} + a_2 \frac{d^2\sigma}{dt^2} = b_o\varepsilon + b_1 \frac{d\varepsilon}{dt} + b_2 \frac{d^2\varepsilon}{dt^2} \tag{10a}$$

if $b_o = 0$.

In this case, the creep function is:

$$\varepsilon = \sigma \left[\frac{1}{E_1} + \frac{t}{3\eta_1} + \frac{1}{E_2} \left(1 - e^{-\frac{E_2}{3\eta_2} t}\right) \right] \tag{11}$$

and the relaxation function:

$$\frac{\sigma}{\sigma_o} = \frac{1}{\rho_2 - \rho_1} \left[\left(\frac{E_2}{3\eta_2} - \rho_1 \right) e^{-\rho_1 t} - \left(\frac{E_2}{3\eta_2} - \rho_2 \right) e^{-\rho_2 t} \right] \tag{12}$$

In equation (12), ρ_1 and ρ_2 are defined by the following equation:

$$3\eta_1 \cdot 3\eta_2 \rho^2 + [E_1(3\eta_1 + 3\eta_2) + E_2\, 3\eta_1]\rho + E_1 E_2 = 0$$

In the theory of linear viscoelasticity, very often the Laplace transform is used for the solution of more complex problems [1].

Although we have shown that in principle the correlation between creep and stress relaxation can be formulated with the aid of the theory of linear viscoelasticity, it can not be expected that real materials behave like idealized elements. The mathematical treatment becomes more complicated if more complex rheological models are taken into consideration. Let us consider a more general case:

$$a_o\sigma + a_1 \frac{d\sigma}{dt} + \cdots + a_n \frac{d^n\sigma}{dt^n} = b_o\varepsilon + b_1 \frac{d\varepsilon}{dt} + \cdots + b_m \frac{d^m\varepsilon}{dt^m} \quad (13)$$

It is demonstrable [2] that equation (13) is expressible by the following integral equations:

$$\varepsilon = \frac{1}{E}\left[\sigma + \int_{-\infty}^{t} K(t-\tau)\sigma(\tau)\, d\tau\right] \quad (14)$$

$$\sigma = E\left[\varepsilon - \int_{-\infty}^{t} \Gamma(t - \tau)\varepsilon(\tau)\, d\tau\right] \quad (15)$$

Although equations (14) and (15) are termed Volterra Integral Equations, Volterra formulated these equations solely upon the basis of Boltzmann's principle of superposition. This principle can be described as follows:

a) If at time $t = 0$ a stress σ_1 is applied to a body, the resulting deformation is expressible by the following formula:

$$\varepsilon_1(t) = \frac{1}{E}\,(1 + \phi(t))\,\sigma_1 \quad (16)$$

If at the same time a higher stress $(\sigma_1 + \sigma_2)$ is applied, the deformation must increase proportionally:

$$\varepsilon_2(t) = \frac{1}{E}\,(1 + \phi(t))(\sigma_1 + \sigma_2) \quad (17)$$

b) If at time $t = 0$ a stress σ_1 is applied, and at time τ an additional stress σ_2, the total deformation must be composed as follows:

$$\varepsilon(t) = \frac{1}{E}\left[1 + \phi(t)\right]\sigma_1 + \frac{1}{E}\left[1 + \phi(t - \tau)\right]\sigma_2 \qquad (18)$$

By the application of Boltzmann's principle of superposition, we can calculate a small change in deformation $d\varepsilon$, which must be proportional to the applied load $\sigma(\tau)$, and to the time $d\tau$ during which the load has been applied:

$$d\varepsilon = \frac{1}{E}\frac{\partial}{\partial\tau}(1 + \phi(t - \tau))\sigma(\tau)\,d\tau \qquad (19)$$

If we replace $\partial/\partial\tau(1 + \phi(t - \tau))$ by the kernel of the integral-equation (14) $K(t - \tau)$, and integrate equation (19) within the limits $-\infty$ and t, the Volterra integral equation (14) is derived.

It must be noted that E and K are not time-dependent in equation (14), and are accordingly inapplicable to an ageing material such as concrete.

The time-dependence for the kernel of equation (14) has been introduced by Arutyunyan [3]. This leads to the following equation for the deformation:

$$\varepsilon = \frac{\sigma}{E(t)} - \int_{\tau_1}^{t} \sigma(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + C(t,\tau)\right]d\tau \qquad (20)$$

where

$$C(t,\tau) = \phi(\tau)\left[1 - e^{-\gamma(t-\tau)}\right]$$

and $\phi(\tau) = C_o + \dfrac{A_1}{\tau}$

C_o, A_1 and γ are constants of the material to be described. The resulting relaxation function is:

$$\frac{\sigma}{\sigma_o} = 1 - \gamma E\left(\frac{A_1}{\tau_1} + C_o\right)\tau_1^p\, e^{r\tau_1}\int_{\tau_1}^{t}\frac{e^{-r\tau}}{\tau^p}\,d\tau \qquad (21)$$

where

$$p = \gamma A_1 E$$

and $$r = \gamma(1 + EC_o)$$

The integral in equation (21) is an incomplete Camma function. Equations (20) and (21) are probably too complicated for most practical calculations.

It can be shown [4] that most of the results of creep experiments are describable by a power function. Figure 2 presents some results obtained for the time-dependence of creep deformation according to [5].

If we choose the following expression for the kernel in equation (14):

$$K(t - \tau) = \frac{c}{(t - \tau)^{\alpha}} \qquad (22)$$

the deformation is given by:

$$\varepsilon = \sigma \left[\frac{1}{E} + \frac{c}{1 + \alpha} t^{1-\alpha}\right] \qquad (23)$$

α for concrete is approximately 0.6 - 0.8. But the relaxation function is then as follows [6]:

$$\frac{\sigma}{\sigma_o} = \sum_{n=0}^{\infty} \frac{(c\Gamma(m)t)^{nm}}{\Gamma(nm + 1)} \qquad (24)$$

This is known as the Mittag-Leffler function and $\Gamma(m)$ is the Camma function.

3. Simplified methods for the calculation of stress relaxation

In the literature, the total deformation is often divided into elastic deformation and creep deformation:

$$\varepsilon = \varepsilon_u + \varepsilon_L = \frac{\sigma}{E(t)} + \frac{\sigma}{E(0)} \phi(t) \qquad (25)$$

This can also be written in the form of a differential equation:

$$\frac{d\varepsilon}{dt} = \frac{1}{E(t)}\frac{d\sigma}{dt} - \frac{\sigma}{E(t)^2}\frac{dE}{dt} + \frac{\sigma}{E(0)}\frac{d\phi}{dt} + \frac{\phi(t)}{E(0)}\frac{d\sigma}{dt} \tag{26}$$

If the second and fourth term on the right hand side of equation (26) are ignored, there is derived the commonly used equation of Dischinger [7]:

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{E_o}\frac{d\phi}{dt} \tag{27}$$

The creep and relaxation functions are easily derived from this equation:

$$\varepsilon = \frac{\sigma}{E_o}(1 + \phi(t)) \tag{28}$$

$$\frac{\sigma}{\sigma_o} = e^{-\frac{E(t)}{E(0)}\phi} \tag{29}$$

Trost [8, 9] applied a differential equation already described in connection with the rheological models (refer to equations (7), (8) and (9))

$$\frac{d\varepsilon}{dt} + \frac{\varepsilon}{\tau} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\tau E}\left(1 + \frac{E}{E_k}\right) \tag{30}$$

where

$$\tau = M_k/E_k$$

Trost employed E instead of E_1, E_k instead of E_2, and η_K instead of 3η.

The creep function is then found to be:

$$\varepsilon = \varepsilon_o \left[1 + \frac{E}{E_k}(1 - e^{-t/\tau})\right]$$

which can be rewritten with $E/E_k = \phi_\infty$:

$$\varepsilon = \varepsilon_o \left[1 + \phi_\infty(1 - e^{-t/\tau})\right] \tag{31}$$

The corresponding relaxation function is:

$$\frac{\sigma}{\sigma_0} = 1 - \frac{\phi_\infty}{1 + \phi_\infty}\left[1 - \exp\left[-(1 + \phi_\infty)\frac{t}{\tau}\right]\right] \qquad (32)$$

The relaxation process approaches the final value more quickly than the creep process, since the exponent in equation (32) is $(1 + \phi)$ times the exponent of equation (31). The final value of the stress relaxation is dependent upon the final value of the creep deformation in the following way [8]:

$$\frac{\sigma_\infty}{\sigma_0} = 1 - \frac{\phi_\infty}{1 + \phi_\infty} = \frac{1}{1 + \phi_\infty} \qquad (33)$$

On the assumption that the change of E as a function of time can be ignored, a comparison will be made of the results obtained by Dischinger and Trost for $\phi_\infty = 2$ and 4.

Dischinger: $\frac{\sigma_\infty}{\sigma_0} = e^{-\phi_\infty} = 0.136$ and 0.018

Trost: $\frac{\sigma_\infty}{\sigma_0} = \frac{1}{1 + \phi_\infty} = 0.33$ and 0.20

In a recent paper, Fuglsang Nielsen [10] has described a modified Dischinger-equation. By taking the creep recovery into consideration, the elastic modulus has been replaced by an effective modulus E', and the creep function ω represents only irrevoverable creep:

$$\frac{d\varepsilon}{dt} = \frac{1}{E'}\left(\frac{d\sigma}{dt} + \sigma\frac{d\omega}{dt}\right) \qquad (34)$$

From this equation, the creep function and relaxation function are found to be:

$$\varepsilon = \frac{\sigma}{E'}(1 + \omega) \tag{35}$$

$$\frac{\sigma}{\sigma_o} = e^{-\omega} \tag{36}$$

In many cases, the discrepancy between the modified Dischinger equation and the formula applied by Trost can be ignored.

Mc Henry [11] has proposed an expression similar to that given by Trost (equations (31) and (32)):

$$\varepsilon = A(1 - e^{-Bt}) \tag{37}$$

and

$$\frac{\sigma}{\sigma_o} = \frac{1}{EA + 1}\left[1 + EA \exp(- Bt(EA + 1))\right] \tag{38}$$

A very simple formula, which combines the stress relaxation and the creep deformation, has been derived by Hansen [12]:

$$\frac{\sigma}{\sigma_o} = \frac{2\varepsilon_o - c}{2\varepsilon_o + c} \tag{39}$$

This formula possesses the advantage that any given creep function c is easily changeable into the corresponding relaxation function. However, for $c = 2\varepsilon_o$, this means for $\phi = 2$, equation (39) leads to zero stress in a specimen. This has certainly not been found to confirm experimental results.

If we define

$$\phi^* = \frac{\varepsilon(t)}{\varepsilon(0)} \quad \text{and} \quad \psi = \frac{\sigma(t)}{\sigma(0)}$$

the Volterra integral equations can be written in a simple way [13]:

$$\int_0^t \phi^*(t - \tau)\psi(\tau)\, d\tau = t \tag{40a}$$

$$\int_0^t \phi^*(\tau)\psi(t - \tau)\, d\tau = t \tag{40b}$$

For concrete, the following equations have very often been employed in the literature to describe the creep and relaxation functions:

$$\phi^*(t) = 1 + \phi_\infty(1 - \sum_1^n e^{-c_i t}) \tag{41}$$

and

$$\psi(t) = \psi_\infty + (1 - \psi_\infty) \sum_1^n e^{-\kappa_i t} \tag{42}$$

The experimental results can be fitted with equations (41) and (42) by the use of an appropriate number for i. When we introduce equations (41) and (42) in equation (40a), the first result obtained in accordance with Trost's paper [14] is

$$\psi_\infty = \frac{1}{1 + \phi_\infty}$$

In [14], it is demonstrated that the creep function can be expressed with the help of values determinable in a relaxation experiment, and also in reverse:

$$\phi^*(t) = 1 + \frac{1 - \phi_\infty}{\psi_\infty} \left[1 - \exp(- \psi_\infty \kappa_1 t) \right] \tag{43}$$

$$\psi(t) = \frac{1}{\phi^*_\infty} + (1 - \frac{1}{\phi^*_\infty}) \exp(- c_1 \phi_\infty t) \tag{44}$$

4. Elementary approach

Finally a description will be given of an elementary approach for the mutual calculation of creep and relaxation functions. In a relaxation experiment, the sum of elastic and creep deformation is kept constant:

$$\varepsilon = \varepsilon_u + \varepsilon_c = \text{const}$$

$$\frac{d\varepsilon_d}{dt} = - \frac{d\varepsilon_c}{dt}$$

$$\frac{d\varepsilon_d}{dt} = \frac{1}{E}\frac{d\sigma}{dt} = - v_c = - F_1(t) \cdot F_2(\sigma) \qquad (45)$$

In equation (45) the only functions allowed are those in which the rate of creep is a function of t, multiplied by a function of σ alone. Equation (45) can be solved in the following way:

$$\int \frac{d\sigma}{F_2(\sigma)} = - E \int F_1(t)\, dt = - E\, \varepsilon_1(t) + c \qquad (46)$$

The table lists some functions frequently employed in connection with the creep of concrete. A more detailed description of these results is given in [15]. They have been inserted in the integral equation (46), and the corresponding relaxation function has been calculated.

Table. Some frequently used creep functions, and the resulting stress relaxation functions based upon equation (46).

	Creep		Stress Relaxation	
	Creep Function	Final Creep Value	Relaxation Function	Final Relaxation Value
A1	$\varepsilon = a t^b \sigma$	∞	$\psi = e^{-Eat^b}$	0
B1	$\varepsilon = \frac{a t}{b+t} \sigma$	aE	$\psi = e^{-\frac{Eat}{b+t}}$	e^{-aE}
C1	$\varepsilon = a(1-e^{-bt})\sigma$	aE	$\psi = e^{-Ea(1-e^{-bt})}$	e^{-aE}
A2	$\varepsilon = a t^b \sigma^n$	∞	$\psi = \sqrt[n-1]{\frac{1}{1+(n-1)\sigma_o^{n-1} E a t^b}}$	0
B2	$\varepsilon = \frac{a t}{b+t} \sigma^n$	$aE^n \varepsilon_o^{n-1}$	$\psi = \sqrt[n-1]{\frac{b+t}{b+t+Eat(n-1)\sigma_o^{n-1}}}$	$\sqrt[n-1]{\frac{1}{1+(n-1)aE^n \varepsilon_o^{n-1}}}$
C2	$\varepsilon = a(1-e^{-bt})\sigma^n$	$aE^n \varepsilon_o^{n-1}$	$\psi = \sqrt[n-1]{\frac{1}{1+(n-1)\sigma^{n-1} E a(1-e^{-bt})}}$	$\sqrt[n-1]{\frac{1}{1+(n-1)aE^n \varepsilon_o^{n-1}}}$
A3	$\varepsilon = a t^b \sinh(c\sigma)$	∞	$\psi = \frac{2}{c\sigma_o} \mathrm{Arth}\left[th(\frac{c\sigma_o}{2}) e^{-Eact^b}\right]$	0
B3	$\varepsilon = \frac{at}{b+t} \sinh(c\sigma)$	$\frac{aE}{\sigma}\sinh(c\sigma)$	$\psi = \frac{2}{c\sigma_o} \mathrm{Arth}\left[th(\frac{c\sigma_o}{2}) e^{-\frac{Eact}{b+t}}\right]$	$\frac{2}{c\sigma_o} \mathrm{Arth}\left[th(\frac{c\sigma_o}{\varepsilon}) e^{-Eac}\right]$
C3	$\varepsilon = a(1-e^{-bt})\sinh(c\sigma)$	$\frac{aE}{\sigma}\sinh(c\sigma)$	$\psi = \frac{2}{c\sigma_o} \mathrm{Arth}\left[th(\frac{c\sigma_o}{2}) e^{-Eca(1-e^{-bt})}\right]$	$\frac{2}{c\sigma_o} \mathrm{Arth}\left[th(\frac{c\sigma_o}{2}) e^{-Eac}\right]$

Figure 3 indicates a comparison of measurements made by Ross [16] with theoretical predictions. Functions B1 of the table were used for this calculation.

In [17], all the parameters of the creep function of type A3 of the table have been experimentally determined for hardened cement paste with a water-cement-ratio of 0.4. Figure 4 illustrates the results obtained of the calculation of stress relaxation, based upon these data, and by the application of equation A3 of the table. It is interesting that the initially applied stress exercises a marked influence upon the time-dependence of the stress relaxation. The results of measurements made in our laboratory with specimens of hardened cement paste, with a water-cement-ratio of 0.3, are illustrated in figure 5. The stress-dependence of the relaxation function can be conframed with this diagramm.

5. Conclusions

It may be said that although a great number of attempts theoretically to describe the relaxation process of concrete have been made, very little experimental work has been described in the literature. For this reason, none of the theoretical approaches can be tested critically. This gap in knowledge should be filled in the near future.

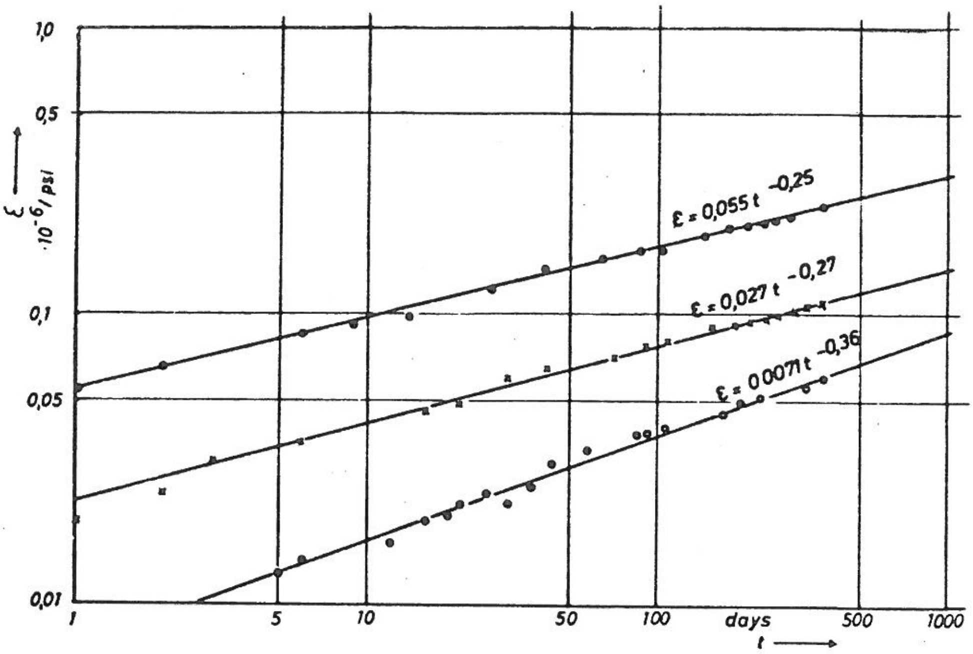

Fig. 2. Results of a creep experiment according to [5]. A power function can be satisfactorily fitted to the experimental results.

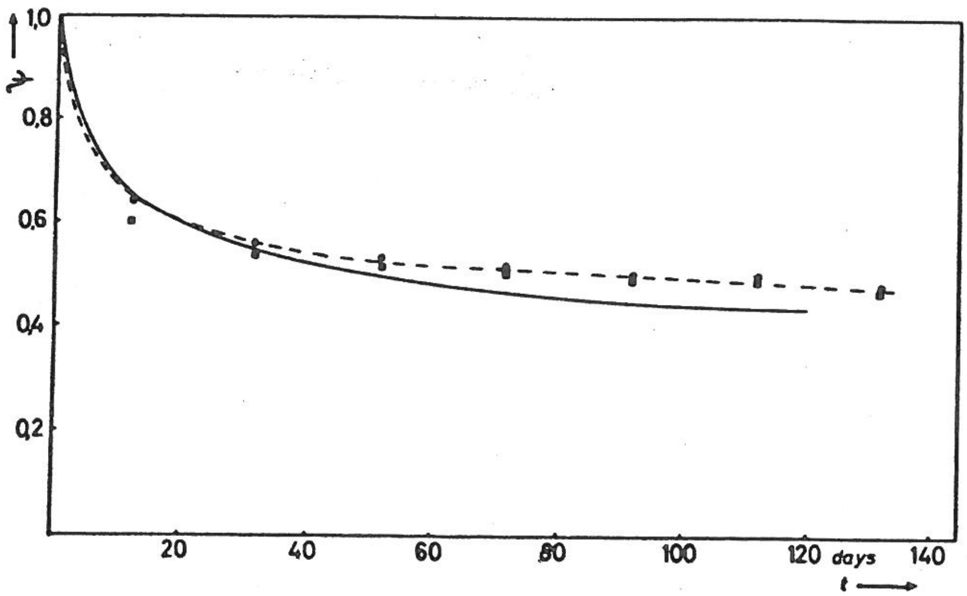

Fig. 3. Experimentally-determined relaxation function according to Ross [16]. With the help of equation B1, the stress relaxation has been calculated from creep data. This function is shown by the line indicated.

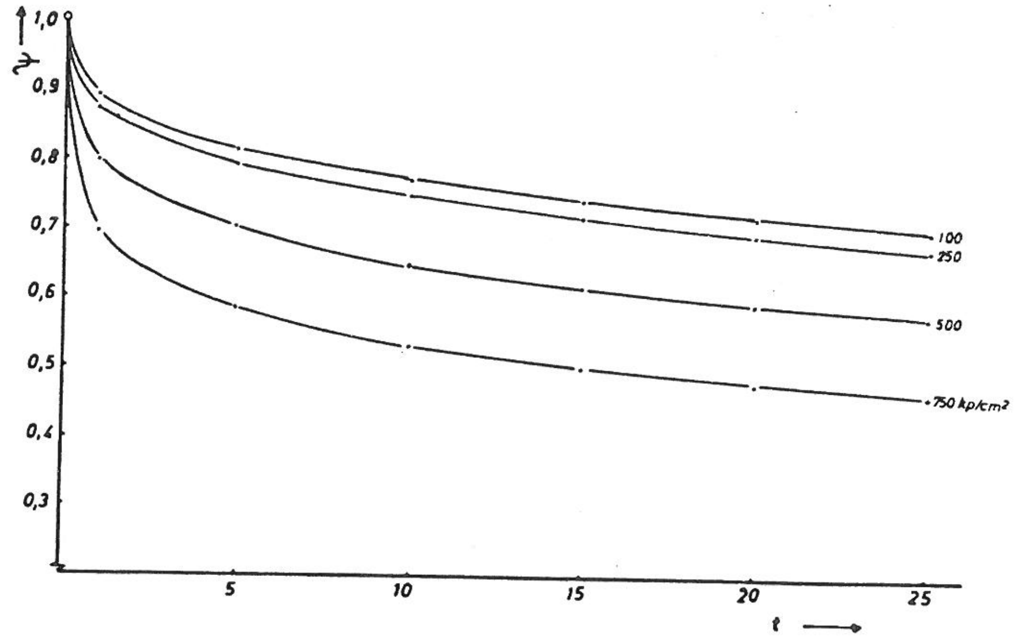

Fig. 4. Equations A3 have been employed for calculation of the stress relaxation from creep data published elsewhere [17].

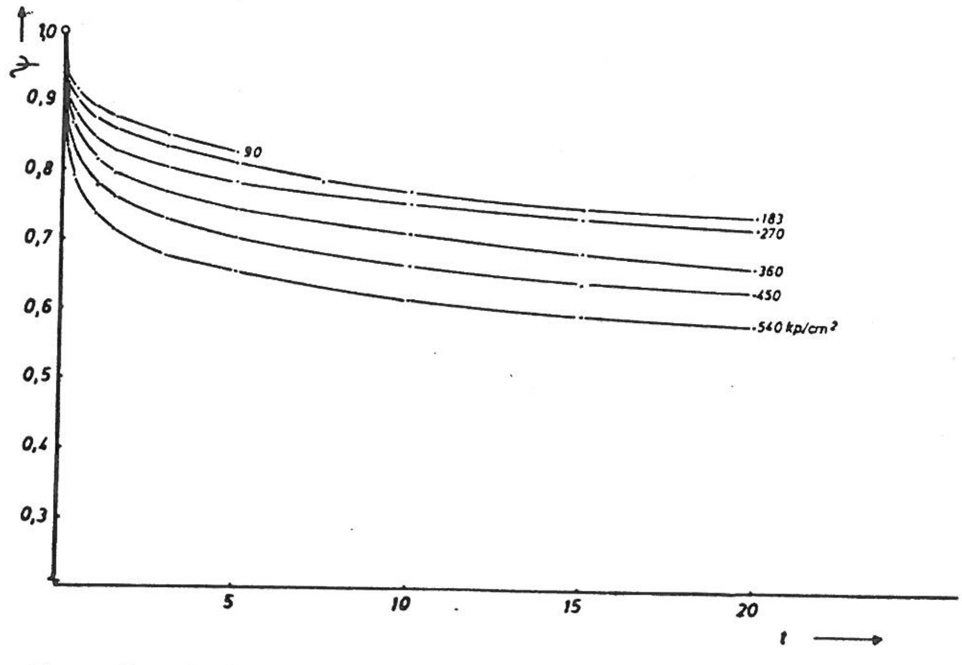

Fig. 5. Experimentally-determined stress relaxation of hardened cement paste as a function of time. The load applied initially was used as a parameter.

[17] Wittmann, F., Physikalische Messungen an Zementstein. Habilitationsarbeit, Technische Universität München 1968.

Folker H. Wittmann, Dr., Priv.-Doz., Technical University of München